Workshop 3.
Inventing Notations

How can a person make an idea visible or keep track of a line of thought? From kindergarten arithmetic to high school calculus, mathematics involves notations—symbols as surrogates for abstract ideas. In Workshop 3, we introduce new but mathematically related investigations: the Pizza problems, beginning with "How many different pizzas can you make by selecting from four toppings?" In this problem, students invent their own notations to represent the different toppings and the combinations that can be made with them. In Workshop 3, teachers learn how to foster and appreciate students' notations for their richness and creativity, and consider some of the possibilities that early work with notation systems might open up for students as they move toward more sophisticated math.

Part 1—Putting It on Paper: Elementary Students Invent Notations

Returning to her classroom in the fall after the summer professional development workshop depicted in Workshop 2, Englewood, New Jersey fourth-grade teacher Blanche Young tries out one of the Pizza activities with her students. Afterwards, she discusses their work with Arthur Powell, the workshop facilitator. Next, fifth-graders from another site in New Brunswick, New Jersey, develop complex and varied notation systems to organize their pizza combinations. Finally, in Colts Neck, New Jersey, a fourth-grader named Brandon comes up with a way to link the Pizza problem to the Towers problem by applying the same notation system to both.

On-Screen Participants: Blanche Young, Englewood Public Schools, Grade 4; Arthur Powell, Rutgers University; and Dr. Alice Alston, Rutgers University. **Student Participants:** Fifth-Graders, New Brunswick Public Schools; and Brandon, Colts Neck Public Schools, Grade 4.

Part 2—Notations Evolve As Students' Thinking Evolves (and Vice Versa)

In the fifth grade, students from the Kenilworth study make a series of leaps into ever more abstract representations of the Pizza problem. They build up to an extension of the problem that is so complex that they can't keep track of their combinations by counting. They find ways to extend their earlier work, inventing notations along the way, and arrive at and defend their solutions.

On-Screen Participants: Dr. Carolyn Maher, Rutgers University; and Dr. Amy Martino, Rutgers University. **Student Participants:** Stephanie, Jeff, Matt, and Michelle, Kenilworth Public Schools, Grade 5.

On-Site Activities and Timeline

60 minutes

Getting Ready

1. Review

a. Share and discuss ideas and experiences with your students concerning the Towers problems.

2. The Three Pizza Problems

In order to get the most out of the video, it is important for teachers to think about their own solutions to the problems—especially the notations, organizational schemes, and strategies that they used. You tried the three Pizza problems as homework for Workshop 2. Familiarity with each of the three problems is important in order to follow what the children in the video are doing. However, in this session, we will focus on Problem Two: Whole Pizzas and Four Toppings. For that reason, after comparing solutions to Problem One: Halves With Two Toppings, you may choose to spend the greater amount of the first hour thinking together about your approaches to Problem Two, leaving discussion of Problem Three for later.

a. Share individual solutions to each of the three problems—one at a time—in small groups and, as time permits, in the whole group. As you do this, pay close attention to what appear to be the important ideas in each solution, especially the different notations developed, the different organizations and strategies employed, and arguments used to justify solutions. Question each other closely about any statements or actions that are unclear to you.

b. After a group is convinced of its solution(s), for each problem: (a) predict—and defend your prediction of—the number of different pizza choices that would result if an additional topping was offered; and (b) ask whether this problem is similar—and in what way—to other problems that you may have encountered.

c. In what ways are the three problems different from and similar to each other?

d. Would you use any or all of these problems with your students? Why or why not? How do you think your students would solve these problems?

On-Site Activities and Timeline

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

The Pizza Problem

Pizza Hut® has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms, and pepperoni. How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possibilities.

Focus Question

What do these three examples—Englewood, New Brunswick, and Colts Neck—have in common in terms of how the notations help students justify their solutions?

Part 2

On-Screen Math Activities

The Pizza Problem With Halves

Capri Pizza has asked us to help design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings can be added to either half of the plain pie or whole pie. How many choices do customers have if they can choose from two different toppings (sausage and pepperoni) that can be placed on either a whole cheese pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

Focus Question

We've seen students spontaneously creating ways of keeping track of their solutions to a problem. What notations are students using to represent their ideas and organize the pizzas?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your homework.

Episode Box One: Exploring Pizza Choices in Classroom Contexts

In the classrooms in Englewood and New Brunswick, we observed children working together in groups on Problem Two: Whole Pizzas and Four Toppings. They developed various ways to represent and keep track of pizza choices and then justify their solutions to others. We also listened to Blanche, the fourth-grade Englewood teacher, as she discussed her concerns for her students.

- What notations and strategies for keeping track of pizza selections did the children use?
- What arguments did the children use to justify their solutions?

Episode Box Two: Making Connections—Brandon

We saw fourth-grader Brandon as he worked on Problem Two with his partner, explained his notation and approach to the problem in an interview, and then connected this problem to Towers Four-High. His initial ideas about connections between the two problems involved surface characteristics. However, as he reorganized his towers into cases and began to recognize the structural identity between the two problems, Brandon mapped the 16 towers in a one-to-one relationship to the 16 pizza selections, based on his "zero-one" notation.

- What mathematical ideas were involved in Brandon's solutions?

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Three: Developing Mathematical Ideas With Pizza Variations

As fifth-graders, the Kenilworth students worked on the series of three pizza problems during two extended classroom sessions approximately one week apart, in the same order they were assigned to you. When solving Problem One in the first session, the students worked in two groups. Each group developed particular notations and grouped the pizza choices by categories, or cases, in order to accurately account for the 10 possible selections.

- What do you notice about the children's problem solving?

In the second session the students first revisited their solutions for Problem One, sharing strategies and ideas. They were then asked to solve Problem Two. Working singly and in small groups, the students developed various notations to keep track of choices and solved the problem quickly, grouping their selections by cases that categorized the pizzas as having: (1) no extra toppings, (2) exactly one topping, (3) two toppings, (4) three toppings, and (5) all four toppings. The extension to two crusts was solved immediately by doubling the 16 pizza choices that they had found. Problem Three proved to be extremely challenging to the students until Matt developed a notation and argument based on the original 16 "whole pizza selections" for Problem Two. He categorized possibilities for pizzas with different halves into cases by holding constant each of the original 16 selections in turn. For each case, the selection held constant became one half of the pizza, while the second half was composed of each of the choices that followed. Matt's final solution was to sum the numbers for each case (15 + 14 + 13 + ... + 2 + 1 = 120) and then add the original 16 whole pizzas (136 possible choices). He credited his approach to Ankur's strategy for Problem One.

- Did you find Matt's argument convincing? Why or why not?

For Next Time

Homework Assignment

Preparing for Workshop 4

1. The assignment for Workshop 4 includes two problems relating to different kinds of towers. These problems will be central to the video in Workshop 4. The first is the classic Towers of Hanoi problem and the second involves Towers Four-High built from Unifix® cubes, this time selecting from three colors.

For each of the two problems, find a way to convince yourself and others that your solution is correct. Keep a record of how you developed and justified each solution, including any written work that you may have done, to share during Workshop 4. After you have completed the problems, ask yourself how they are similar to and different from each other, and what similarities there might be between these problems and other problems that you have encountered.

Problem One: Towers of Hanoi

Legend has it that a group of Eastern monks are the keepers of three towers on which sit 100 golden rings. Originally, all 100 rings were stacked on one tower with each ring smaller than the one beneath. The monks' task was to move the entire stack of rings from this first tower to the third tower one at a time, using the second tower as necessary, but never placing a larger ring on top of a smaller one. The legend was that once all 100 rings have been successfully moved, the world will come to an end.

Dr. Robert B. Davis presented this problem to the Kenilworth sixth-graders in the following way:

> "There is an order of monks in the city of Hanoi and they were concerned about when the world is going to end. So they made a puzzle like this."
>
> Dr. Davis held up a Towers of Hanoi model—a rectangular base with three dowels attached perpendicular to that base. Eight rings were stacked on one of the dowels, with each of the eight smaller than the one immediately below it.

For Next Time

Homework Assignment, cont'd.

> "The monks spend all of their time working to solve the puzzle. When they have it done, that is supposed to be when the world ends. I thought it might be interesting to figure out when the world is going to end, so that we would know it, too. So the question is this: How many 'moves' (the smallest possible number) would it take to complete the task?
>
> "Let's agree on what the rules are: (1) You can move exactly one disk at a time. (2) You can never put a bigger disk on top of a smaller one.
>
> "So how many 'moves' would it take to complete the task?"

Problem Two: Ankur's Challenge With Towers Four-High
In the 10th grade, Ankur presented the following problem to the group:

> Find as many towers as possible that are four cubes high if you can select from three colors and there must be at least one of each color in each tower. Build a solution for this problem, selecting from three colors of Unifix® cubes.
>
> How do you know that you have found all the possibilities? Convince your peers that you have found all the possibilities—no more and no fewer.

2. Use the Pizza problems with your students. Observe their activity carefully and take notes about how they approach the problem, paying particular attention to how their notations, strategies, and approaches are similar to and different from the ways that you thought about the problem and what you observed in the video. Collect interesting samples of notation from your students for your journal and to share in Workshop 4.

Reading Assignment

The reading assignment can be found in the Appendix of this guide.

Maher, C. A. "The Nature of Learning." In *Can Teachers Help Children Make Convincing Arguments? A Glimpse Into the Process,* 21-34. Universidade Santa Ursula: Rio de Janeiro, Brazil, 1998.

Maher, C. A., and Martino, A. M. "Brandon's Proof and Isomorphism." In *Can Teachers Help Children Make Convincing Arguments? A Glimpse Into the Process,* 77-101. Universidade Santa Ursula: Rio de Janeiro, Brazil, 1998.

Notes