Workshop 2.
Are You Convinced?

The simple question, "Can you convince me?" is key to the mathematical success of the students in the Rutgers long-term study. This program introduces the idea of proof as one of the key ideas in mathematics. Delving into the mathematics of the Towers problem, we'll look at how two kinds of proof—proof by cases and proof by induction—naturally grow out of the need to justify and convince others. The teacher plays a critical role by using several kinds of questions to help students move toward successful justification of their answers.

Part 1—Teachers Building Proofs

Englewood, New Jersey, is a district in transition toward a more thoughtful approach to teaching and learning in mathematics. In a professional development workshop led by Arthur Powell, a researcher in the Kenilworth study, teachers in the district come up with creative and mathematically sound justifications for their solutions to the Towers problem. These solutions are similar in many ways to the solutions offered by the students, and in fact, many of the approaches arrived at independently by these teachers will appear again in the series.

On-Screen Participants: Dr. Joyce Baynes, Superintendent, Englewood Public Schools; Arthur Powell, Rutgers University; and Englewood, New Jersey Teachers, Grades K-8.

Part 2—Students Building Proofs

Returning to the Kenilworth study, we examine the research footage in detail as the students justify and convince the researchers and each other that they have found a way to determine the number of combinations that can be made in towers n high. That students—at this early age—can come up with mathematically sound proofs has important implications for teachers at all grade levels.

On-Screen Participants: Dr. Carolyn Maher, Rutgers University. **Student Participants**: Stephanie, Jeff, Michelle, and Milin, Kenilworth Public Schools, Grades 3 and 4.

On-Site Activities and Timeline

60 minutes

Getting Ready

1. Review

a. Discuss reflections and ideas you have had since Workshop 1.

b. Share experiences from your students' exploration of the Shirts and Pants problem.

2. Towers Four-High

a. Share individual solutions to the Towers Four-High problem from your homework in small groups and, if time permits, in the total group. As you do this, pay close attention to what appear to be important ideas, strategies, and arguments in each solution. Question each other closely about any statements or actions that are unclear.

b. After your group is convinced of its solution(s) for towers four cubes high, predict—and defend the prediction of—(a) the number of different towers three cubes high when selecting from two colors; and (b) the number of different towers five cubes high when selecting from two colors. Is it possible from your solution for towers four-high to figure out a general rule to find the number of towers of any height?

c. Is this a problem you might use with your students? How do you think that your students would solve this problem?

60 minutes

Watch the Workshop Video

Part 1

On-Screen Math Activities

Towers

Build all possible towers that are five (or four, or three, or *n*) cubes high by selecting from plastic cubes in two colors. Provide a convincing argument that all possible arrangements have been found.

Focus Question

We have seen teachers presenting a number of carefully constructed arguments for finding all of the combinations of towers four-high, when selecting from two colors. Which arguments are convincing? Why?

Part 2

On-Screen Math Activities

Towers, cont'd.

See description above.

Focus Question

What are some similarities and/or differences in the mathematical reasoning by the teachers and the students that you observed?

On-Site Activities and Timeline

30 minutes

Going Further

Please address as many of these questions as possible during your allotted time and consider the remainder as part of your homework.

Episode Box One: Teachers and Towers—Patterns and Cases

As the teachers made their presentations, Group One displayed a table with five entries to show the number of towers four cubes high in each of five sets of towers, where all the towers in a set were built using the same number of blue cubes in each tower. As they showed the drawing of the 16 towers in their solution, arranged according to these five sets (or cases), they explained that they had identified the towers in each set by "trial and error" methods.

Group Two explained that they also used cases, but different than Group One, to identify their 16 towers. Their cases were defined as sets of towers that included from one to four cubes of a color "together."

The third group also based their solution on cases; in this instance—five.

- Discuss the organization of the cases by these three groups.

Math Note: Solutions described here and in later workshops include a number of important ideas and strategies that may lead to formal mathematical proofs. Here, they include:

1. Patterns

Patterns frequently recognized and used to construct new towers and to develop a convincing solution include: (a) "opposites"—pairs of towers with the color of each cube in the first tower replaced by the alternate color to form the partner tower; (b) "flips"—pairs of towers with the color pattern from top to bottom of the first tower constructed from bottom to top to form the partner tower; (c) "staircases"—sets of towers beginning with a single bottom cube of one color in the first tower, two bottom cubes of that color in the second tower, and so on until the fourth tower is completely made up of that color; (d) "elevators"—sets of towers with one cube of the first color and three of the second, with the single-colored cube placed in each of the four possible positions.

2. Proof by Cases

The set of towers is separated into groups, or "cases," so that every tower will be included in exactly one case. A convincing argument is then made, case by case, that all possible towers for that case have been found.

On-Site Activities and Timeline

Going Further, cont'd.

Episode Box Two: Teachers and Towers—Doubling

The final presentation by the teachers was made by the group that we observed earlier in the video as they were working out a justification for their towers four-high. Their conclusions were based on solutions they had found for towers one-, two-, and three-high. This group arranged the towers of each height in a manner that showed how each new tower was built from a tower that was one cube shorter. They pointed out that for each successive height, the number of towers of that height doubled from the number found for the height one cube shorter.

- Explain why you did or did not find this to be a convincing argument.

Math Note: This solution includes ideas that lead to mathematical proof by induction. The statement is made that for towers that are one cube in height there can be only two possible towers: one of each color. From this beginning, the inductive argument is developed that if we know the number of towers n cubes high, then there must be twice that number for towers of height $n + 1$, since each of the n towers would account for two towers of height $n + 1$, built by adding one of either color to the original base tower. Tree representations may be built that show the development of this logical argument from towers one cube high to the 16 towers that are four cubes high.

Episode Box Three: Children and the Towers Problems

In the video, we saw the Kenilworth children working together in classroom sessions in grades three and four, building solutions to problems about towers. We later observed some of the same children, in interviews, working hard—even struggling and arguing with each other—to develop convincing justifications about their solutions and what they had discovered about the towers.

- What can we say about the students' methods of solving these problems and what happened to them over time?

Math Note: The mathematical term "combinatorics" is mentioned throughout the videotapes to describe particular kinds of problems. Combinatorics problems involve the mathematics of systematic counting based on strategies, such as pattern recognition and grouping. These ideas are important for children as they develop operational skills with whole numbers, basic understanding of probability and discrete mathematics, algebraic concepts such as variables, and overall ideas about justification and generalization. The Principles and Standards for School Mathematics (2000) calls for investigations involving combinatorics to be included throughout school mathematics.

For Next Time

Homework Assignment

Preparing for Workshop 3

1. Study the Episode Boxes on the previous pages. Reflect on the questions and record your reactions and ideas in your journal.

2. Try the following three Pizza problems. For each of the problems, find a way to convince yourself and others that you have found all possible pizzas and that you have no duplicates. Keep a record of how you developed each solution, including any written work that you may have done to develop your solutions and justifications, to share during Workshop 3.

After you have completed the three problems, ask yourself how they are similar to and different from each other and what similarities there may be between any of these problems and the Towers problem.

Pizza Problem One: Halves With Two Toppings

A local pizza shop has asked us to help them keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings can be added to either half of the plain pie or the whole pie. How many possible choices for pizza do customers have if they can select from two different toppings (sausage and pepperoni) that can be placed on either the whole cheese pizza or half a cheese pizza? List all possible selections.

Pizza Problem Two: Whole Pizzas and Four Toppings

The pizza shop has asked for help again. This time they are offering a basic cheese pizza with tomato sauce. A customer can then select from the following toppings to add to the whole basic pizza: peppers, sausage, mushrooms, and pepperoni. How many different choices for pizza does a customer have? List all the possible selections.

The local pizza shop was so pleased with our help that they have asked us to continue the work. Remember that they offer a cheese pizza with tomato sauce and that the customer can then select from four toppings: peppers, sausage, mushrooms, and pepperoni. The shop owner now offers a choice of crusts: regular (thin) or Sicilian (thick). How many choices for pizza does a customer now have? List all possible combinations.

Pizza Problem Three: Halves and Four Toppings

At customer request, the pizza shop has agreed to fill orders with different choices for each half of a pizza. Remember that they offer a basic cheese pizza with tomato sauce. A customer can then select from four toppings: peppers, sausage, mushrooms, and pepperoni. There is a choice of crusts: regular (thin) and Sicilian (thick). How many choices for pizza does a customer now have?

3. Use the Towers problems with your students. Observe their activity carefully and take notes on how they approach the problem, paying particular attention to how their strategies and approaches are similar to and different from the ways in which you thought about the problem and what you observed in the video.

For Next Time

Reading Assignment

The reading assignment can be found in the Appendix of this guide.

Maher, C. A. and Martino, A. M. “Conditions for Conceptual Change: From Pattern Recognition to Theory Posing.” In H. Mansfield and N. H. Pateman (Eds.), *Young Children and Mathematics: Concepts and Their Representation.* Sydney, Australia: Australian Association of Mathematics Teachers, 1997.

Maher, C. A. and Martino, A. M. “Young Children Invent Methods of Proof: The Gang of Four.” In P. Nesher, L. P. Steffe, P. Cobb, B. Greer, and J. Goldin (Eds.), *Theories of Mathematical Learning,* 431-447. Mahwah, NJ: Lawrence E. Erlbaum Associates, 1996.