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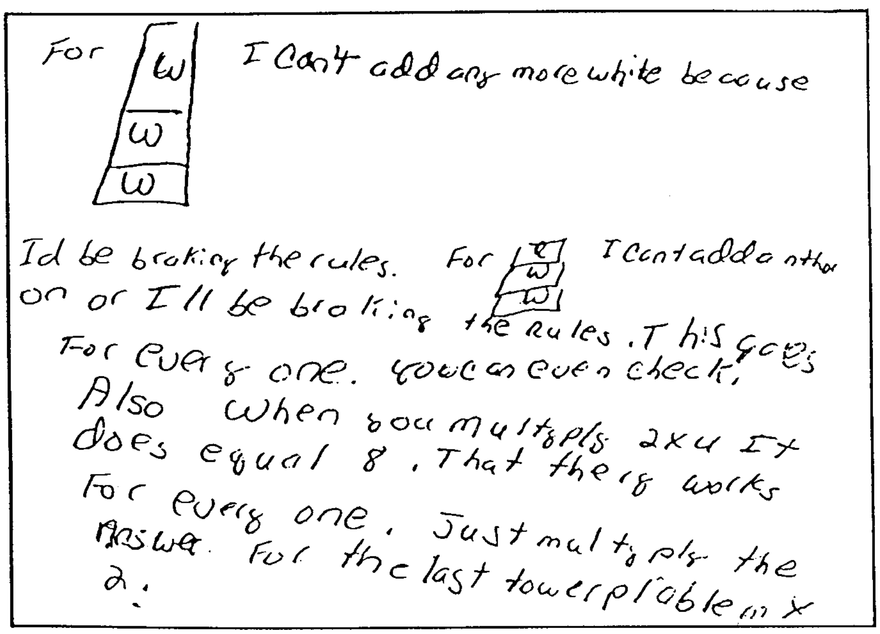
Can teachers help Children make convincing arguments?

A glimpse into the process



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volume 5

Chapter Six:

Brandon's proof and isomorphism

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Brandon and Justin Building Towers

(November 17, 1992)

When Brandon and his partner, Justin, were first challenged to solve the *Tower Problem* for towers 4-cubes tall, they generated new towers by trial and error, then built a "partner" tower for each new tower. Sometimes they made an "opposite" partner and other times an "upside-down" partner. (See Figure 1).

The boys, seated as partners, nevertheless worked separately building different "partner" groupings. Eventually, they recognized that using two different "partner".groupings sometimes resulted in producing duplicate towers. (See Figure 1). They adjusted for this difficulty by discarding the "upside-down" pairing strategy and grouped solely by "opposite" pairs. This procedure resulted in eight pairs of "opposite" towers. (See Figure 2).

Brandon Writes About the Tower Problem

(December 17, 1992)

In a classroom written assessment one month later, the children were asked to find all possible towers 3-tall when selecting from two colors of plastic cubes, and to individually justify in writing that they had accounted for all possibilities. Brandon produced a written justification where he again used the organizing strategy of pairs of "opposites" as a form of argument to show all possibilities. (See Figure 3)

Brandon and Colin Working on the Pizza Problem

(March 11, 1993)

Four months after the administration of the *Tower Problem*, Brandon worked with a new partner, Colin, to find all possible pizza combinations when selecting from four toppings. He used the heuristic of constructing a chart and developed a notation for representing pizzas when he was asked to make pizzas selectiong from four toppings. He assigned the digit "zero" to represent the absence of a topping and the digit "one" to represent its presence. This notation, like other children's use of checklists and numerical codes ultimately aided Brandon in developing a method of proof (Maher, Martino, & Alston, 1993).

Once Brandon developed a notation for recording his combinations, he began to generate pizzas by guess and check methods and recorded each different combination in his chart. Within this chart the beginning formation of local organizations was apparent. For example, three pizzas with exactly three toppings were reported together. (See Figure 4.)

> **Teacher**: What are you doing here, Brandon?
>
> **Brandon**: Making a graph just like Colin. I put peppers, sausage, mushrooms and pepperoni down and have them like 1, 0, 1, 0 and put... and make a graph.

A portion of Brandon's initial graph is pictured in Figure 4.

> **Teacher**: What does that mean 1, 0, 1, 0?
>
> **Brandon**: Well instead of using [writing] like pepper down or sausage... you just gotta put like a "1" for yes it's going in, and a "0" for no it's not.
>
> **Teacher**: So then this first pizza has what on it?
>
> **Brandon**: Peppers and mushrooms... and then you could have all [toppings] or zero [toppings]... just a plain.
>
> **Teacher**: What would you write for an all plain one [a pizza with no toppings]?
>
> **Brandon**: 0, 0, 0, 0.
>
> **Teacher**: Are you doing something like that too, Colin?
>
> **Colin**: Yeah.

Brandon: Yeah, he's doing it similar.

Colin had also created a chart, but was using check marks to represent the inclusion of a topping and leaving blank the column(s) in which a topping was excluded. (See Figure 5)

Brandon decided to make a new chart and reorganized his solution using a series of groupings: two toppings, all toppings, three toppings and no toppings. Brandon's second organization is displayed in Figure 6. When he was challenged by his teacher to develop an argument which would show that every possible pizza had been considered, Brandon reorganized his pizzas into an arrangement which took the form of a proof by cases. Brandon's third attempt at building a chart began with the symbols (1, 1, 0, 0), which he quickly changed to (0, 0, 0, 0), as indicated in Figure 7.

Brandon: Colin, you can just have one of each. Colin... you could do this... [He records the four possible pizzas with exactly one topping which appear as entries 2 through 5 on Figure 7].

Colin: I have 15 so far... I think I have... I have 15.

Brandon: Fifiteen?

Colin: Yeah, see look there's the half, the pepper alone, the the pepper and sausage, then the pepper and mushroom, then the pepper and pepperoni, and I put sausage...

Brandon:Oh Colin... Colin... I just noticed... ya know how you're going like that [points to the columns from peppers towards the right] then you could start from the middle and work your way down like that [points to the columns from mushroom towards the right]. That's how I did something like this...

Thus, Brandon generated all the two-topping pizzas by combining the toppings in the first column [peppers] with each of the three remaining toppings [mushroom, sausage and pepperoni], then combining the topping in the second column [mushrooms] with each of the remaining two toppings [sausage and pepperoni] and the topping in the third column [sausage] with the remaining one topping [pepperoni]. Note how Brandon attempted to match his notation to the notation of his partner.

Colin: Oh yeah, the middle...

Brandon: Wait... I'm gonna try... 0, 0, 1, 1, ya got 0, 0, 1, 1? [Brandon's notation] Look... blank, blank 1, 1. ["Blank" is Colin's method of indicating the absence of a topping].

Colin: Uh Hmm. I have every one.

Brandon: That's it.

Colin: Fifteen

Brandon: Okay... fifteen?

Brandon counted the eleven combinations he had represented in his new chart. These included: the pizzas with no toppings; then, exactly one topping; and then, exactly two toppings. He then predicted that there would be more than fifteen possibilities since the number of choices for zero, one, and two toppings [1 choice, 4 choice and 6 choices, respectively] continued to increase.

Brandon: Eleven. And I didn't even get to three [He refers to including the pizzas with three toppings.]... Colin, I didn't even do three's and I'm into 11 [combinations] so there must be more than 15 [possibilities]. Okay, watch... [Brandon continues to record combinations for exactly three and four toppings arriving at a total of 16 possibilities.]

Towards the conclusion of the activity, Brandon and Colin compared and checked their sixteen solutions. Colin, in trying to make sense of Brandon's representation, attempted to translate Brandon's code.

Colin: Peppers, sausage and mushroom?

Brandon: Peppers, sausage and mushroom?

Colin: You have two of the same [duplicate pizzas].

Brandon: Oh no, no I checked the wrong one...

Colin: Yeah, that's pepperoni and that's pepperoni and mushroom... [Colin points to Brandon's paper.] and sausage and mushroom and pepperoni... wait, do you have a double? [Colin searches Brandon's chart.]

Notice that in building their arguments the boys refined their notational systems. The framework that evolved was then used as a structure to justify that all combinations of toppings had been considered.

In the process of searching for all possible pizza choices when selecting from four pizzas toppings, Brandon had invented a code to represent the presence of specific toppings on a pizza. He constructed a chart with four columns, one column for each of the four toppings, and represented the toppings as follows: "P" for pepper, "M" for mushroom, "S" for sausage and "peponi" for pepperoni. To represent a pizza, Brandon decided to place either a 1 or a 0 in each of the topping columns to indicate whether or not a topping was present on the pizza under consideration.

Thus, when he placed a "0" in the peppers column it meant that the topping peppers was not included on the pizza under consideration, whereas a "1" in the sausage column meant that the topping sausage was included on the pizza. Using this system of coding, Brandon developed a *proof by cases* for all possible pizzas selecting from four toppings. His five cases included the following: (1) zero toppings [a plain pizza with cheese and tomato sauce], (2) one topping, (3) two toppings, (4) three toppings, and (5) four toppings.

An Interview with Brandon
(April 5, 1993)

When interviewed about his method of proof three weeks after working on the Pizza Problem, Brandon retrieved his former notation of zeros and ones and began to organize his possibilities. Initially, he organized his work by starting with the pizza with no toppings, then making a pizza with toppings A, followed by a pizza with toppings A and B, followed by a pizza with toppings A, B and C. After Brandon had listed fourteen pizzas on his chart, he decided to draw another graph in which all pizzas with n toppings (n = 0, 1, 2, 3, or 4) were grouped together. He was then able to prove that he had accounted for all possible cases, and was further able to demonstrate how he had accounted for all possible pizzas within each case. (See Figures 8a & 8b)

> **Brandon**: Nothing on a pizza... [places a "0" in each of the four topping columns of his chart to represent a pizza with no toppings] this you can have one pepper on a pizza with nothing else... [places a "1" in the pepper column of his chart and zeros in the other three columns to represent a pizza with peppers only] one mushroom on a pizza with nothing else... [places a "1" in the mushroom column and zeros in the other three columns] and you could have a couple of sausages on the pizza with nothing else... [places a "1" in the sausage column and zeros in the other three columns] maybe a couple of pepperonis... [places a "1" in the pepperoni column and zeros in the other three columns].

Brandon continued to organize his pizzas by cases and recorded each with a four digit sequence of zeros and ones. Here he considered the group of six pizzas which he called "two's". (See entries 6 through 11 in Figures 8a & 8b).

Brandon: And if you don't want any of that [pizzas with no toppings or one topping] you can start getting fancy, and start going to "two's", a pepperoni [sic, pepper] and mushroom and nothing else, then pepperoni [sic, pepper] and sausage and nothing else, then pepper and pepperoni and nothing else... you could have mushroom and sausage and nothing else.

Int: How did you know to go to mushroom now? [See entry 9 in Figure 8a.]

Brandon: Why didn't I use pepper anymore?

Int: Yeah.

Brandon: Cause I already used pepper there [with mushroom]... it's mushroom and pepper [refers to entry 6 on Figure 8a]. And if I put a "1" down for mushroom and pepper that would be the same thing... So each time you go three, two, one...

Brandon explained that he could combine the first pizza topping [peppers, in this case] with the three remaining toppings, the second topping [mushrooms] with the two remaining toppings and the third topping [sausage] with the one remaining topping [pepperoni].

Brandon: You get three choices with the first one [the topping peppers], then with the second one... with mushrooms you only get two choices cause there's only sausage and pepperoni, then with sausage you can only do pepperoni.

As the interviewer probed further, Brandon reaffirmed his strategy for finding all possible pizzas with two toppings. In the process of explaining his method, Brandon articulated an understanding of distinction between a selection situation in which the order of the items in each subset would not affect the total number of choices [combinations] and a situation in which order would affect the total number of choices [permutations]. He recognized that the Pizza Problem was a situation dealing with combinations and illustrated his point by explaining that having an airplane and a car was the same as having a car and airplane, regardless of the order of the two items.

Int: What I don't understand is... when you move to [the] mushrooms [column] why can you put it with sausage and pepperoni, but you can't put it with peppers?

Brandon: Cause that would be a same thing... if I do that and I put a "1" there [in the peppers column] I've already got pepperoni [sic, peppers] sausage... that would be a same thing. It's just like saying you have an airplane and a car... saying you have a car and an aiplane... it's still the same thing.

Making a Connection

Using his system of coding and organization by the number of toppings, Brandon was able to provide an inclusive organization by cases for his toppings and account for all possible pizzas. When the interviewer asked Brandon if the problem reminded him of any others that he had worked on in the past, Brandon recalled the *Towers Problem* given four months earlier.

Int: In any way does it remind you of any of the problems we've done?

Brandon: It kind of a little reminds me of the blocks [towers] cause my partner and I... whoever it was... I think it was Colin... did them in order like... yellow, red, yellow, red and then switch around and do the "opposite"... Red, yellow, red, yellow. It's kind of like what you do here [He refers to the case groupings for pizzas.] Just do it in groups... that's what we did with the blocks... how many ways can you make towers... how many ways can you make pizzas... the same problem, sort of.

Int: The same kind of thing? Do you remember how many towers there were?

Brandon: I think I can remember...

Brandon took red and yellow cubes and quickly reproduced eight towers of height four as four pairs of "opposites". Recall that an organization by "opposites" was his strategy for the *Tower Problem* four months earlier when he worked with his partner Justin and three months earlier on his written assessment. It is interesting that Brandon remembered his prior organization for towers, but did not recall with whom he had worked to develop that organization.

When asked to defend how the organization by "opposites" was an effective method to account for all possible towers, Brandon paused for about one minute. He appeared to be studying the chart he had invented to organize his pizza combinations. Brandon attempted to make another chart, this time for towers. He began with two columns and labeled the red and yellow to represent the colors of the cubes. However, he quickly discarded this representation and returned to the plastic cubes and began to build new arrangements.

Brandon built one pair of "opposite" towers with three cubes of one color and one cube of the opposite color, and then built the remaining towers with three yellow cubes and one red cube. He did this by moving the one red cube up one position at a time starting with the tower with the red cube in the bottom position. He then built the "opposite" set of towers with one yellow cube and three red cubes by moving the yellow cube up one tower position at a time. (See Figure 9).

Brandon completed his task of organizing the eight towers in Figure 9 into two staircases, each with four towers. He then verbalized his recognition of a connection between the tower and pizza problems, and enthusiastically shared with the interviewer his "new" discovery.

> **Brandon:** It's kind of like the pizza problem... like this would be the one's group. [referring to the eight towers of height four with three cubes of one color and one cube of the opposite color]
>
> **Int:** Let's see.
>
> **Brandon:** Oh, yeah... I see! This now... this is like the one's group [referring to the eight towers in Figure 9].
> You only have one of the opposite color in these [towers with exactly one red or one yellow cube]...
> This isn't how I did it, but I just noticed this...
>
> **Int:** This is very interesting.
>
> **Brandon:** I just noticed it... then you would have... that would be the one's group, you only have one in there.
> Then... in pizzas [holds up the all red and all yellow towers] this would be like the "whole" group or "all" group... save that for last. Now you have the "two's" group... twos [He referred to the six towers of height four with two red cubes and two yellow cubes]. You have two of the opposite color in there.

For Brandon's new organization of towers, he moved from 8 sets of paired "opposites" to 3 sets of towers: (1) the "all/whole group" which included the two towers with cubes all the same color, (2) the "one's group" which contained the eight towers with one cube of one color and three cubes of the "opposite" color, and (3) the "two's group" comprised of six towers of height four with two cubes of each color. He eagerly talked with the interviewer as he worked. His new organization placed "opposite" towers within the same set. (See Figure 10).

Int: Okay, so you said these [six towers of height four with two red and two yellow cubes] all have two... two yellows or two reds?

Brandon: They had two of each color... they must to be in the "two's" group... if they had three of a color and one of the opposite color they'd be in the "one's" [group]... and you won't have any "three's" group [for towers]...

Int: Why not?

Brandon: Cause for the "three's" group you have three yellows and one red... that would be the same [pointing to the eight towers in his "one's" group].
A "three" group is like that group [pointing to the "one's" group] cause three of the opposite color... So a "three" group would be the same as a "one" group.

Int: Now if I wanted to call all this [the eight towers in the "one's" group] a three's group...

Brandon: Yeah, you could call it a "one" or a "three" group.

Int: Why?

Brandon: You could call it a "three" group cause it has three [cubes] of one color and one [cube of the] opposite [color], or you could call it a "one's" group because it has one of the opposite color. You can call that "three" or "one" it doesn't matter which.

To determine whether Brandon would recognize the structural isomorphism between finding all pizzas selecting from four toppings and finding all towers four cubes tall selecting from two colors, the interviewer asked Brandon to focus his attention on one color within the towers and then to reconsider his groupings.

Brandon decided to place his attention on the yellow cubes within each tower, and changed his groupings of towers from three cases to five cases: (1) towers with no yellow cubes, (2) towers with one yellow cube, (3) towers with two yellow cubes, (4) towers with three yellow cubes and (5) towers with four yellow cubes. (See Figure 11.)

It was at this point in the interview that Brandon, enthusiastically, expressed that the group of four towers with exactly one yellow cube were like the four pizzas with one topping in his chart, and placed each tower on top of its corresponding pizza on the chart. He explained how the red cubes in each tower corresponded to the "zero's" on his pizza chart and how the yellow cubes in each tower corresponded to the "one's" on his chart. He then confidently proceeded to match each of the sixteen towers to each of the sixteen pizzas represented on his chart.

> **Int:** Now, what if we put our focus on say the yellow [cubes]... if we're looking at these eight towers here and we're looking at yellow, and we're looking for the "one's", what would be a ones tower?
>
> **Brandon:** A "one's" yellow tower?
>
> **Int:** Yeah.

Brandon picked up a tower with exactly one yellow cube in the second from the top position and responded "That would be a "ones" yellow tower and a three red tower." The interviewer asked what else would be a "one yellow" tower. Brandon's response was to build a staircase pattern with the four towers with exactly one yellow cube. As he arranged the staircase, he said: **"It's like the pizza problem... you work your way down.** Like pepperoni", pointing to the tower with exactly one yellow cube in the top position. He continued: "mushroom", pointing to the second from the top position, and "pepper", indicating the bottom position. He then picked up his pizza chart and said: "You start with zero."

When the interviewer asked, "What would the "zero one" [pizza with no toppings] look like if we're looking at yellow [cubes]?", Brandon immediately responded by knocking down all sixteen towers and pushed them to the side of the table. The interviewer replied, "I don't understand". Brandon smiled and said, "Blank". Suggesting that such a tower / pizza correspondence could not be represented.

As Brandon realigned the four towers with exactly one yellow cube, the interviewer asked. "Tell me again how this is like the pizzas?". Brandon explained, "You have the one pepperoni... since we're looking at yellow [cubes], the yellow cube would be 1 and the red would be 0". He compared the red cubes in the towers to the digit "0". He continued, "You could have one pepper... then it's like stairs if I draw a line down here like this... it's sort of like here you'd have one pepperoni, one mushroom, one sausage and one pepper." As he spoke, Brandon pointed to the one yellow cube in each of the four towers. Thus, he had noted the similarity between the four towers of height four with exactly one yellow cube and entries two through five in his pizza chart, and then traced a diagonal line on the chart to show the similarity. (See Figure 8a.)

> **Int:** Is what you're saying to me that a yellow cube is like a number "1" in you chart?
>
> **Brandon:** Yeah... if you're focusing on red, then red would be number "1". Now you would work with the "two yellow group"...

He then took each tower with exactly two yellow cubes and placed it above the corresponding zero / one code on the pizza chart establishing a one-to-one correspondence between each of the six "two-topping" pizzas and the six towers of height four with exactly two yellow cubes.

Brandon: Right here... you would have the yellow stand for one. So it would be yellow for one, red zero, yellow one, red zero.

Int: I see.

Brandon: That'd be another one.

Int: So this would come next.

Brandon: ... yellow, red, red, yellow. Yellow, red, red, yellow.

Int: I see. So what would this pizza look like? This one? [referring to a tower with yellow cubes in the top and bottom tower positions]

Brandon: That would be pepperoni and... pepper and pepperoni.

Brandon then compared the four possible pizzas with three toppings to the four towers with exactly three yellow cubes. Brandon took the tower with exactly four yellow cubes and called it the "pizza with everything" or four toppings, and concluded that "since the red [cubes] would stand for zero this [the tower with four red cubes] would be the "zero guy", or the tower with no yellow cubes.

Int: You know what I'm wondering? We have this guy left, right? [the tower with four red cubes]

Brandon: Yeah, cause we're not focusing on red.

Int: If we had to call him a name though...

Brandon: Oh, oh... this would be the zero. Oh yeah. Since the red would stand for zero, this would be the zero guy.

Int: This is neat. This is really neat, Brandon.

Brandon: Finally found out what the [all] red [tower] would be... the zero guy.

Brandon also articulated the symmetry of his argument and explained that if he had chosen to focus on the red cubes within each tower instead of the yellow cubes, the yellow cubes in each tower would correspond to the "zeros" on his pizza chart and the red cubes in each tower would correspond to the "ones". He then reorganized his *proof by cases* for towers to show towers with (1) no red cubes, (2) one red cube, (3) two red cubes, (4) three red cubes and (5) four red cubes.

Int: Could we have done it the other way around? Could we have focused on red and gotten it to work the same way?

Brandon: Same way. Now the red [cubes] would be one and the yellow [cubes] would be zero.

Brandon's insight, in recognizing the structural relationships between the tower and pizza problems, came about by his active building and rebuilding of representations, over an extended period of time, in situations that encouraged communication and thoughtfulness.

For each problem task, Brandon and his classmates worked for approximately one and one half hours, over two consecutive days. The children's explorations included two problems, towers and pizzas. The charge was to find all possibilities and to convince us that all were found. The children were not expected to be immediately successful. A four-month time period was allotted for the investigations, so that earlier ideas could be revisited, connections could be discovered, and extensions be made. The teacher did not tell the children whether their solutions were correct or incorrect. The intent was clearly to give this responsibility to the children whose task was to decide, based on the arguments they put forth.

Whether some, or indeed all of these conditions, contributed to Brandon's success is not certain. What is clear, however, is that Brandon was doing mathematics under these conditions and that he was doing it very well.

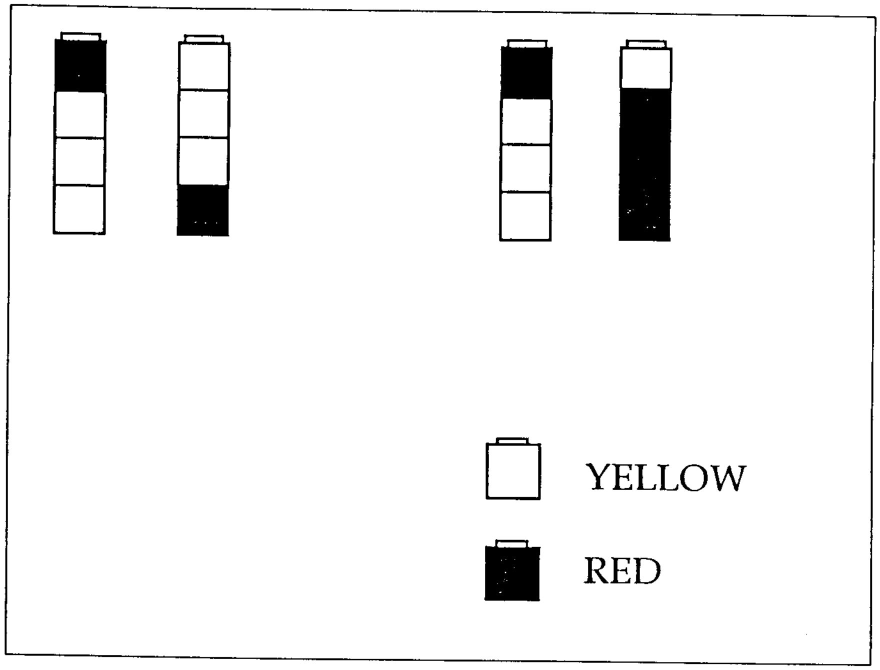


FIG. 1

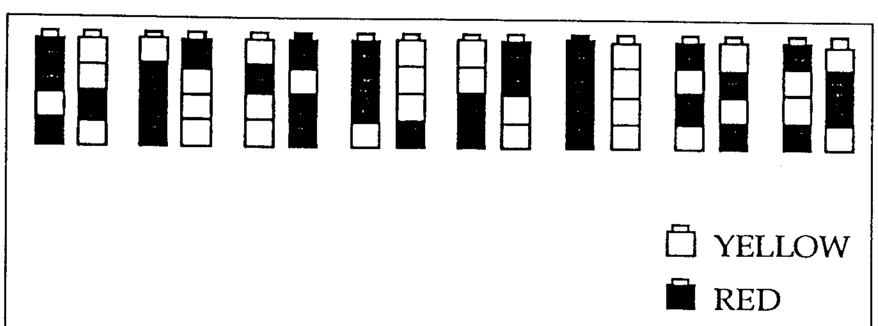
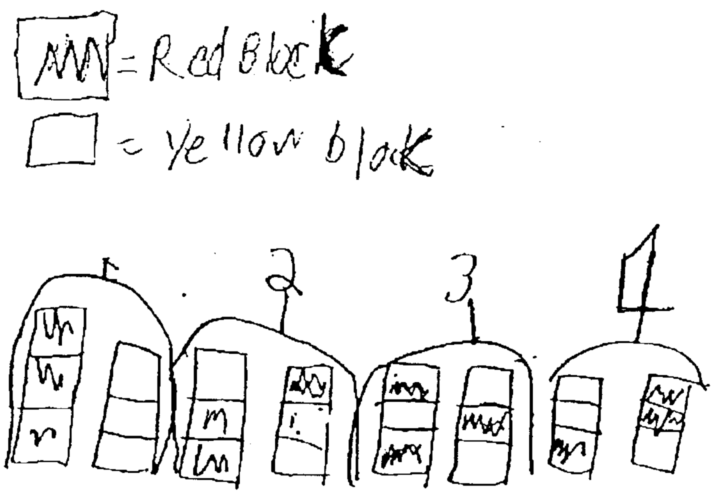


FIG. 2

Please write a letter to someone who has never done this activity. Describe all of the different towers that can be built that are three cubes tall, when you have two colors available to work with. Explain in your letter why you are <u>sure</u> that you have made every possible tower and that there are no duplicates.



Dear ?

I'm in Mrs Zallys' math class and we are bilding. towers. There are 4 different oppisit pares.

FIG. 3

P	S	M	Peponi	[sic]
1	0	1	0	
1	1	1	1	
1	1	0	1	
1	1	1	0	
0	1	1	1	
1	0	0	1	

FIG. 4

	P	S	M	Peponi
1st	✓	✓		
2nd. pizza	✓			
3rd. pizza	✓		✓	
4th. pizza	✓			✓
5.		✓	✓	
6.		✓		✓
7.		✓		
8.			✓	
9.			✓	✓
10.				✓
11.	✓	✓	✓	✓
12.	✓	✓	✓	
13.		✓	✓	✓
14.	✓		✓	✓
15.				

FIG. 5

P	S	M	Peponi [sic]
1	0	0	1
1	1	0	0
1	0	1	0
1	1	1	1
1	0	1	1
1	1	1	0
1	1	0	1
0	0	0	0

FIG. 6

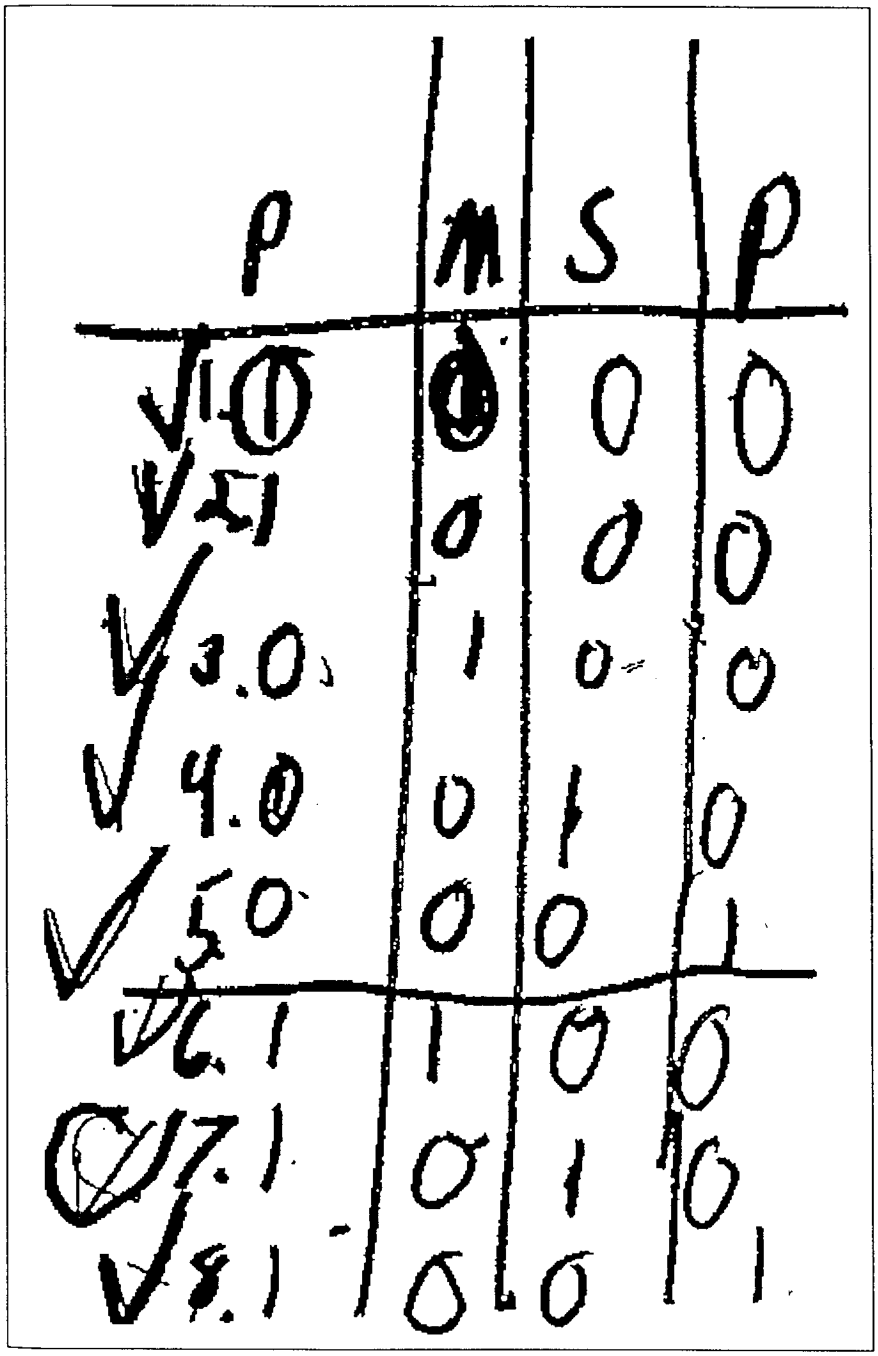


FIG. 7

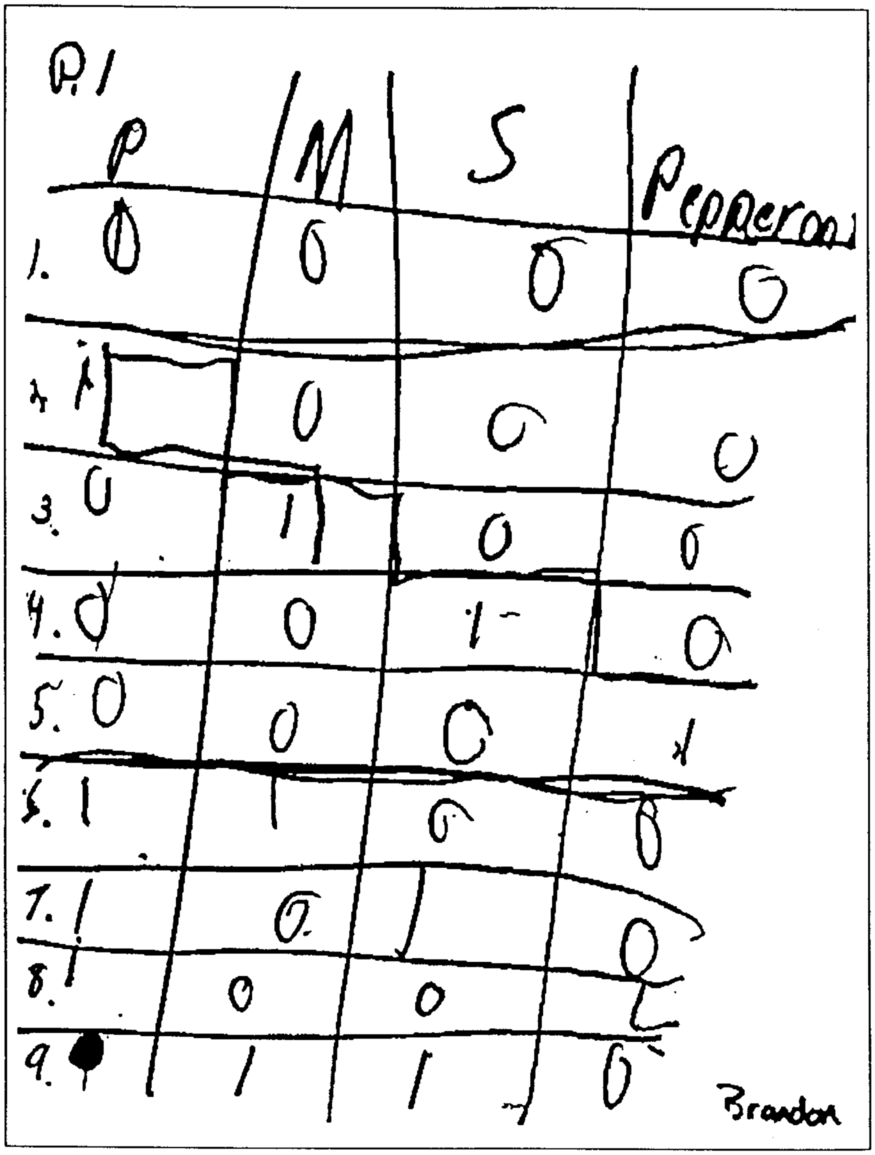


FIG. 8a

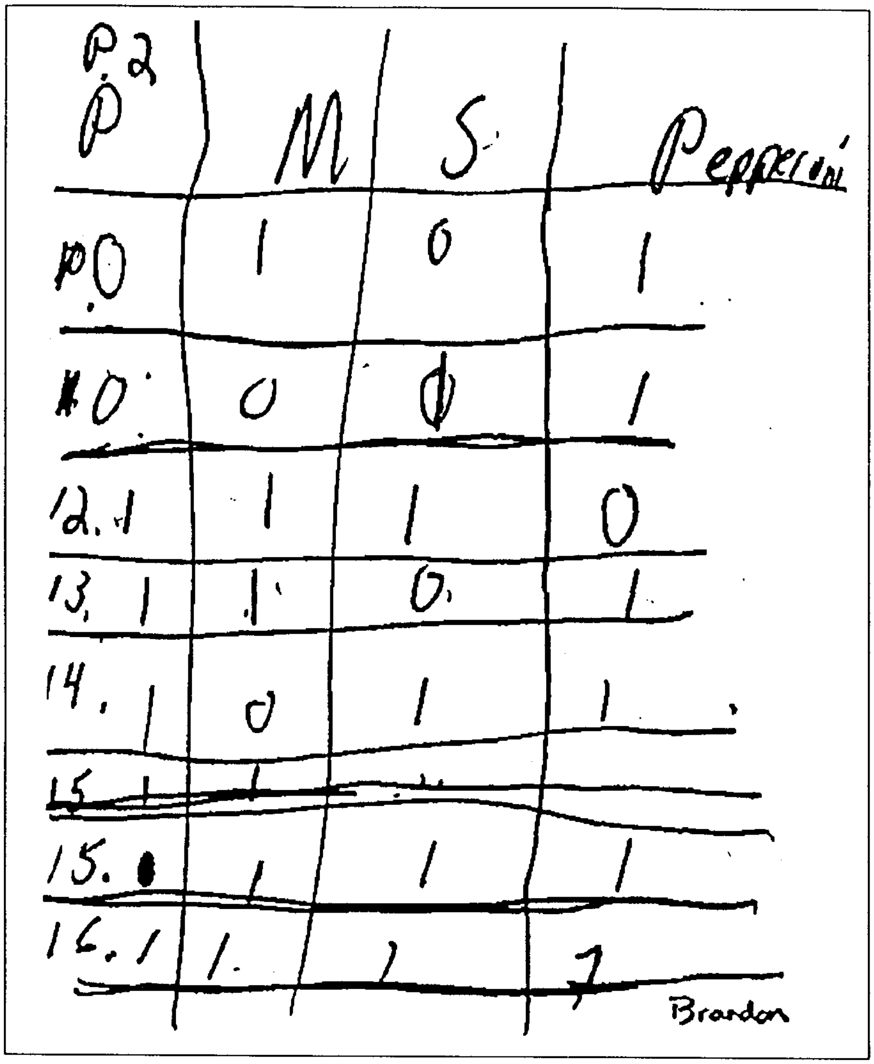


FIG. 8b

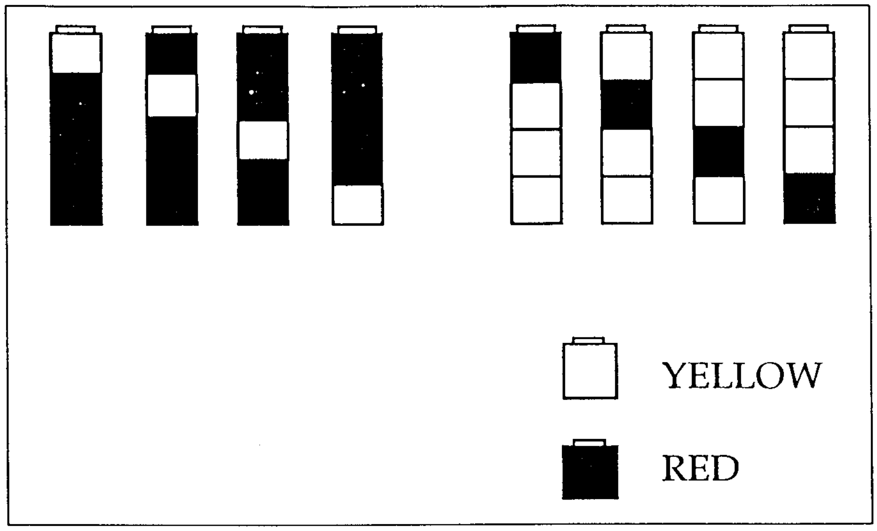


FIG. 9

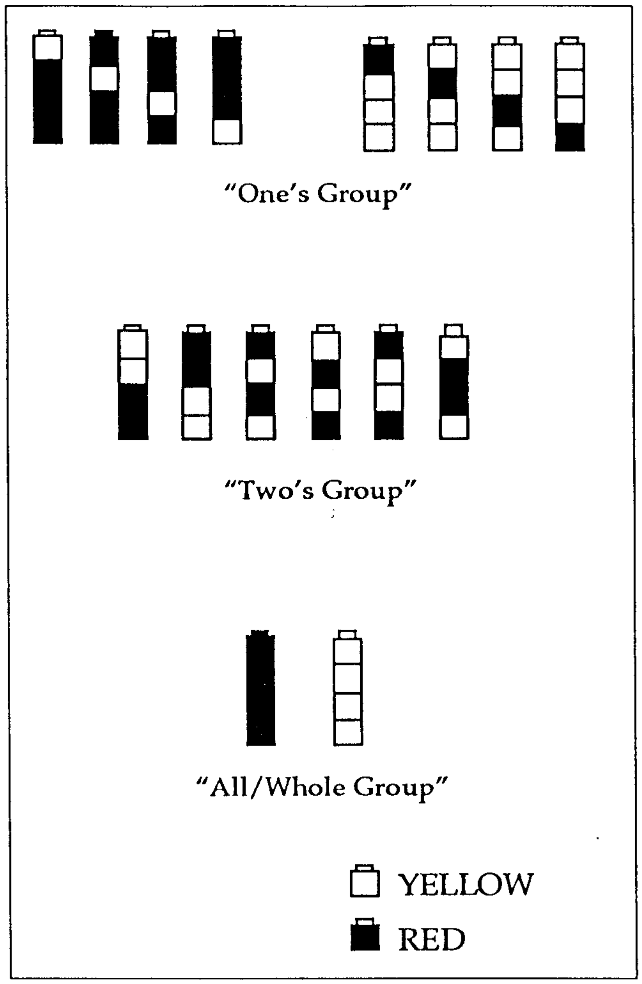


FIG. 10

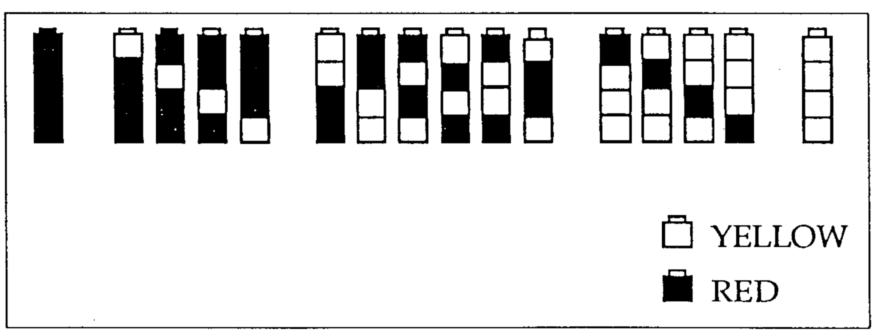


FIG. 11