CONDITIONS FOR CONCEPTUAL CHANGE: FROM PATTERN RECOGNITION TO THEORY POSING[3]

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Introduction

> Two, four, eight, sixteen… that's weird! Look! 2 × 2 is 4, 4 × 2 is 8 and 8 × 2 is 16. It goes like a pattern! You have the 2 × 2 equals the 4, the 4 × 2 equals the 8 and the 8 × 2 equals the 16.
>
> *Stephanie, Grade 4, March 6, 1992.*

Stephanie, a nine year old, studied the models of towers that she built with stacks of Unifix cubes of varying heights using two colours.[4] She recognised the sequence of total number of towers from each height-classification as a *pattern*.

> What humans do with the language of mathematics is to describe patterns.
>
> *Lynn Steen, 1990, p.8.*

A few minutes later, when questioned by her teacher about her work, Stephanie gave a rule to describe a method for generating the towers:

> All you have to do is take the last number that you had and multiply by two.
>
> *Stephanie, Grade 4, March 6, 1992.*

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4 There were two towers 1-cube tall; four towers 2-cubes tall; eight towers 3-cubes tall and sixteen towers 4-cubes tall.

Steen (1990) describes mathematics as:

> …an exploratory science that seeks to understand every kind of pattern — patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns. (p.8)

A few days later, in a small-group discussion with the teacher and three other students, Stephanie accounted for finding all possible towers that were 3-cubes tall when selecting from two colours by inventing a 'proof by cases'.[5] Stephanie was not aware at this time that the 'doubling pattern' could also generate a display of all possible towers and be used to account for all of them. She commented on the limitation of the 'doubling method' and noted that the difficulty was in producing the tower arrangements, which she referred to as 'patterns'.

> The hard part is to make the patterns. Like you could… from now we know how to just like … oh, you could give us a problem like how many in ten and we could just go… I **know** the answer. I figured it out … One thousand twenty four.
>
> *Stephanie, Grade 4, March 10, 1992.*

According to Steen, in order to grow mathematically,

> …children must be exposed to a rich variety of patterns appropriate to their own lives through which they can see variety, regularity, and interconnections. (p.8)

Eight months later, in checking the total number of towers 4-cubes tall that could be generated to solve another problem, Stephanie recalled her 'doubling method' and referred to it as a 'theory'. In this instance, there was a discrepancy between the total number of towers produced by the 'doubling procedure' and the total number of towers which Stephanie could actually find.

> Well a couple of us figured out a **theory** because we used to see a pattern forming. If you multiply the last problem by two, you get the answer for the next problem. But you have to get all the answers. … I don't know what happened! Because I am **positive** it [the doubling rule] works.
>
> *Stephanie, Grade 5, February 26, 1992.*

Efraim Fischbein writes about intuition:

> In a world of potentially misleading uncertainties our practical decisions cannot rely only upon indirect inferences, on theoretically based suppositions. We feel the fundamental need 'to see' with our mind, as we see with our eyes. (1987, p.7.)

Later, during this same episode, Stephanie recognised how the 'doubling method' could be used to generate all possible towers. Smiling, an exuberant Stephanie exclaimed:

> Yes! I knew it! I knew it! I knew it! … I told him all along. I was right… so we're going to talk to the class about this?
>
> *Stephanie, Grade 5, February 26, 1992.*

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3 Stephanie's arrangement by cases included none of a color, exactly one of a color, exactly two of a color "stuck together", exactly three of a color, and exactly two of a color "separated". For more detail, see Maher & Martino, in press.

Continuing on intuition, Fischbein writes:

> An intuition is a theory, it implies an extrapolation beyond the directly accessible information. (1987, p.13.)

A story about Stephanie

During a one-year period, Stephanie frequently was videotaped doing mathematics in school. Sometimes she worked with a partner or a small group; other times she shared her thinking during a whole-class session or during a task-based interview. Within these various settings, particular attention is given to tracing how Stephanie developed an understanding of **argument** by *mathematical induction.*

Also, during this period, Stephanie had developed an **argument** using *proof by cases* to justify a solution to a problem she had been working on for some time (Maher & Martino, in press). The focus of this chapter is to study Stephanie's thinking as a fourth-grader, beginning with the March 6, 1992 interview when Stephanie first recognised a 'doubling pattern', to the February 26, 1993 fifth-grade class activity in which she presented her argument by mathematical induction. Twelve episodes from Stephanie's video portfolio are presented to support Stephanie's conceptual change. Our story begins with Stephanie's attempt to 'make sense of' the mathematical pattern that she discovered in grade 4.

Students' learning and proof

Balacheff (1991), in addressing the benefits and limits of social interaction in learning mathematics, points out that a goal in most curricula is to teach mathematical proof. He further indicates that achieving this goal is very difficult citing particular work by Gaud and Guichard (1984), Bell (1976), Senk (1985), and Usiskin (1982).

Hanna (1991) calls attention to the realities of mathematical practice and challenges earlier views of mathematics as reasoning by deduction that culminates in formal systems. She writes:

> Mathematicians agree, furthermore, that when a proof is valid by virtue of its form only, without regard to its content, it is likely to add very little to an understanding of its subject and ironically may not even be very convincing. (p.55)

Citing as support the writings of Davis (1986) who sees various roles for proof, Tymoczko (1986) who advocates for the convincing benefits of informal proofs, and Kitcher (1984) who challenges the tenet that the basis of mathematical knowledge is proof, Hanna continues:

> In these more recent views, a proof is an argument needed to validate a statement, an argument that may assume several different forms as long as it is convincing. (p.56)

Balacheff and Hanna discuss the role of social interaction in determining the acceptance of a convincing argument, warning of the possibilities for misunderstanding and of the limits as well as the benefits.

On teaching, Gila Hanna writes:

> That reasoning is a pedagogical issue at all bespeaks a conviction that the learning of mathematics is a dynamic rather than static process, in which students progress towards deeper level of insight and skill. (1991, p.60.)

In reviewing the research on mathematical proof, Alibert and Thomas (1991) give attention to studies that show how understanding of proofs may be better communicated within settings that consider the social aspects of the activity. Reference is made to the work of Fischbein (1982), Movshovitz-Hadar (1988) and Tall (1979) whose interest has been on the teaching of proofs in a meaningful way. Their work is mainly concerned with older students' introduction to proof in mathematics courses.

David Tall (1991) in making a distinction 'between the way in which ideas are built cognitively and the way in which they are arranged and presented in a deductive order' warns of the danger of 'presenting mathematical theory as a sequence of definitions, theorems and proofs' (p.xiv).

Interest in how mathematicians develop advanced mathematical ideas but fail to pass on to students the thinking processes that enabled them to do so resulted in attention by mathematicians and mathematics educators to improve the situation, according to Tall. Although much of the attention has been given to advanced mathematical thinking of older students, more recent reform movements have shifted attention onto reasoning and justification for younger children (NCTM, 1989; NCTM 1991; National Research Council, 1991).

How, then, do young children build up their ideas cognitively? Why would a child want to write a justification? How do children learn to write a mathematical proof? What particular mathematical explorations would lend themselves to inventions by a young child?

The study

In presenting a story about a ten-year-old child, Stephanie, and how she worked with several other classmates to make sense of a rule that she discovered, the process by which children invent arguments to support their thinking and continue to be engaged in the problem is studied.

Content domain

The mathematical content, combinatorics, is explored initially with a problem task in which students are asked to build towers of particular heights when they can select from plastic cubes of two colours. Freudenthal (1991) cites the area of **combinatorics** as an area that is most appropriate for reinvention. He writes:

> Starting with numerical paradigms, guessing general relations, experiencing and satisfying needs for good definitions and convincing proofs, encountering mathematical induction thanks to these efforts, and using mathematical induction, first instinctively, then intentionally, and eventually in a more or less formally verbalised manner — all this together appears to be a most efficient course in reinvention. (p.53.)

Setting

The research takes place in the Kenilworth Public Schools. The partnership began as a teacher development project in mathematics from 1984–1987. Since 1988 the classrooms in Kenilworth have become laboratories where researchers at Rutgers University and teachers in the Kenilworth Public Schools study children's mathematical thinking.

In this setting, children are observed building solutions, constructing models, comparing models, developing arguments, discussing their ideas and negotiating their conflicts as they work on problematic situations which have been posed. The teacher both facilitates discussion and probes student thinking through questioning which is related to students' constructions (Martino & Maher, 1994). Thus, instruction is frequently guided by the questions raised by the students that often provide the stimulus for re-examination of a problem, or exploration of a problem extension. Closure on a topic is deferred until a later time when students are prepared to resolve their questions. For this reason, students often explore a problem over an extended interval which can span days, weeks, months or years.

Methods and procedures

All mathematics lessons are videotaped with three or four cameras to capture students working on problem tasks and their interactions with other students and the classroom teacher. Sometimes students are interviewed by the teacher either individually or in a small-group following the problem task. These interview sessions are videotaped with two cameras: one camera focusses on the student(s) and the other records the written work produced by the student(s).

Data source

Data consist of the following: (1) videotapes of small-group and whole-class problem-solving sessions and individual student interviews; (2) accompanying student written work; (3) researcher notes recorded on-site; and (4) student written assessments related to a problem activity.

Analysis

The videotapes are transcribed, coded and verified for accuracy by at least two graduate-student researchers (see Note 1). Transcripts of the videotapes are analysed and episodes are identified to form the contents of a video portfolio.

Stephanie's video portfolio

Episodes over a one year interval were identified that provided insight into the conditions which contributed to Stephanie's conceptual change beginning with pattern recognition to building an intuitive understanding of an argument by induction. Specifically, in building groups of towers of varying heights using Unifix cubes, we focus on Stephanie's discovery of 'take the last number that you had and multiply by two' [6] to her recognition of the generation of the total number of towers (n+1) tall from towers n tall.

Background

In the Fall of grade 3 as a whole class problem-solving activity, Stephanie and her classmates were given Unifix cubes and asked to build all possible towers 4-cubes tall when selecting from red and blue cubes (Martino, 1992). The following year, in February of grade 4, students were asked to generate all towers which were 5-cubes tall selecting from two colours.

In the month following the building of towers 5-cubes tall, individual interviews were conducted with students. Stephanie was interviewed three times in the period of four weeks as she pursued the building of an argument to convince others that she had accounted for all possibilities. Based on an exploration of towers of varying heights from 1-cube tall to 6-cubes tall, Stephanie progressed from initially using 'guess and check' methods to later organising groups of towers according to certain patterns, called 'local organisations.'[7] In so doing, she came to appreciate the necessity for reorganising some local patterns in order to account for all tower possibilities and to avoid duplication of combinations. These activities led her to construct the global scheme that produced a 'proof by cases' (Maher & Martino, under review).

The first episode in this study begins with Stephanie's invention of a 'proof by cases' that took place on March 6, 1992 during her third interview involving towers. In this session, she was able to account for all possible towers 4-cubes tall when selecting from two colours. She also noticed a 'doubling pattern' for the total number of towers of consecutive heights. The session is followed by eleven subsequent episodes which trace Stephanie's development of an intuitive argument by mathematical induction. The twelve episodes from Stephanie's video portfolio summarised in *Table 1* are analysed in detail in the results section.

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6 Videotaped interview with Stephanie on March 6, 1992. Stephanie conjectured that the total number of towers n cubes tall was equal to the total number of towers (n-1) cubes tall times two.

7 These would be described as use of a variety of pattern organizers rather than one over-all organizational scheme. A limitation of local organizations is the potential formation of duplicate tower arrangements. For more detail, see Maher & Martino, under review.

Episode #	Date Recorded	Description
Episode 1	March 6 1992 75 min.	Interview: Stephanie discovered that the number of towers (selecting from two colours) doubles each time the height of the towers is increased by one cube. She used this pattern to calculate towers of heights 6 and 10.
Episode 2	March 10 1992 45 min.	Small-group discussion: Stephanie provided her justification for having found all possible towers 3-cubes tall, a version of proof by cases, to three classmates. She also acknowledged the existence of the 'doubling pattern', but did not attempt to explain how it worked.
Episode 3	May 15 1992 30 min.	Interview: Stephanie was asked to consider how the 'doubling rule' could be used to generate all possible towers. She was introduced to a tree diagram by the interviewer, but did not choose to represent her tower arrangements this way.
Episode 4	June 15 1992 40 min.	Paired Written Assessment: Stephanie worked with Milin to provide a written explanation to an absent classmate for finding all possible towers 3-tall and to account for all possibilities. She used the 'doubling pattern' to check that all towers were found.
Episode 5	October 25 1992 40 min.	Individual Written Assessment: Stephanie, working alone, provided a written justification for accounting for all possible towers 3-cubes tall using a proof by cases. Again she used the 'doubling rule' to check her work.
Episode 6	February 26 1993 120 min.	Classroom Activity: Stephanie, working with Matt, attempted to find all possible towers, 4-cubes tall, when selecting from plastic cubes in two colours. In this instance her 'doubling pattern' did not match the number of towers which she was able to find, causing her to question the validity of the 'doubling rule'.
Episode 7	February 26 1993 120 min.	Classroom Activity: Stephanie regained confidence in her 'doubling rule' and now questioned whether she had accounted for all possible tower arrangements.
Episode 8	February 26 1993 120 min.	Classroom Activity: Stephanie shared her ideas with other classmates. In so doing, she expressed frustration with the incompatibility of the 'doubling rule' and her result.
Episode 9	February 26 1993 120 min.	Classroom Activity: Stephanie attempted, unsuccessfully, to connect the doubling rule' to the generation of all possible towers. Consequently, when she tried to explain her method to another classmate, it didn't make sense.
Episode 10	February 26 1993 120 min.	Classroom Activity: Matt joined the discussion and presented the 'tree organisation' that Milin and Michelle had built. Stephanie listened to Matt's explanation, appearing, for the first time to understand how the 'doubling pattern' worked.
Episode 11	February 26 1993 120 min.	Classroom Activity: Stephanie interrupted Matt's explanation and continued to build taller towers. Excitedly, she became aware of the connection between the 'doubling pattern' and the generation of towers by use of a 'tree of towers'.
Episode 12	February 26 1993 120 min.	Whole-Class Discussion: Stephanie shared the 'tree of towers' with the entire class. Her tone of voice and demeanour indicated a feeling of confidence and delight.

Table 1. Twelve episodes which trace Stephanie's movement from pattern recognition to theory posing.

Results

Episode 1: Stephanie discovers the 'doubling pattern'

During an interview which took place on March 6, 1992 in grade 4, Stephanie discussed her solution to building all possible towers 4-cubes tall when selecting from black and white plastic cubes. She justified her solution of 16 towers using a form of **proof by cases**. She then used her new organisation by 'cases' to find all possible towers of heights 3-cubes tall, 2-cubes tall and 1-cube tall when selecting from two colours. After building each solution with plastic cubes, Stephanie spontaneously discovered a pattern between the total number of towers of different heights.

Stephanie:	Two [towers 1-cube tall], four [towers 2-cubes tall], eight [towers 3-cubes tall], sixteen [towers 4-cubes tall]… that's weird! Look! 2 × 2 is 4, 4 × 2 is 8 and 8 × 2 is 16. It goes like a pattern! You have the 2 × 2 equals the 4, the 4 × 2 equals the 8 and the 8 × 2 equals the 16.
Teacher:	I wonder why?
Stephanie:	Well it turns out that every [total] number [for towers of a particular height] is even.
Teacher:	Now if this is a pattern, what would you guess for towers of 5?
Stephanie:	If I had to guess…32.
Teacher:	You would guess 32?
Stephanie:	Yeah, 32.
Teacher:	Is there a reason why that would work?
Stephanie:	I remember when we did the towers of 5 in class… 32!
Teacher:	I wonder why this works? Let's look at what we've got here…
Stephanie:	Because you double it!

The teacher pursued Stephanie's recognition of this pattern and encouraged her to speak more about 'doubling' the total number of towers. In the process of sharing her thinking, Stephanie generated a rule or 'method' based upon this pattern.

Teacher:	So what do you have here?
Stephanie:	A method.
Teacher:	What's the method?
Stephanie:	All you have to do is take the last number that you had and multiply by two.
Teacher:	Are you convinced that that's always gonna work?
Stephanie:	Yes.

Stephanie then used her new 'method' to calculate the total number of towers 6-cubes tall which she determined was 64 towers. She was asked by the teacher to think about how the method worked. She was also asked to consider how many towers could be built that were 10-cubes tall. *Figure 1* shows Stephanie's calculation of the total number of towers, 1024.

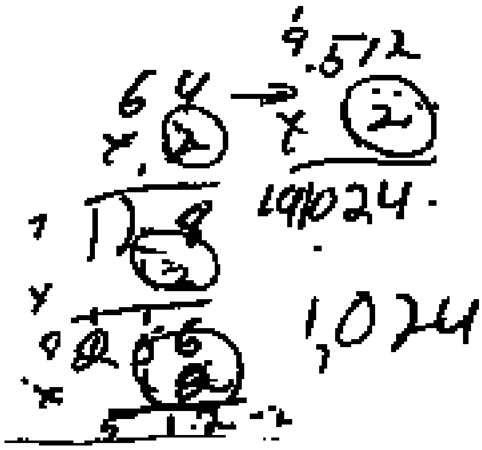


Figure 1: Stephanie's use of her 'doubling method' to calculate the total number of towers from 6-cubes tall to 10-cubes tall.

Significance

This opportunity to re-examine her strategies for justifying the total number of towers of different heights allowed Stephanie to discover another potentially powerful method for obtaining towers. She had discovered that the total number of towers doubled each time the height of the towers was increased by one cube. She then posited a rule from this pattern and used this rule to calculate towers 6-cubes tall and 10-cubes tall.

Episode 2: Justifying towers 3-cubes tall

During a conversation with the teacher and three of her classmates on March 10, 1992, Stephanie was again challenged to think about the generation of the total number of towers of different heights. It was in this session that she tried to convince a classmate, Jeff, that she could account for all possible towers 3-cubes tall by producing an argument by cases. Another student, Milin, provided a different argument. He used a 'tree' organisation for finding all towers 3-cubes tall when selecting from red and blue cubes (Alston & Maher, 1993). [8] During this session, Jeff, Michelle and Stephanie each acknowledged the 'doubling pattern' for the total number of towers. Jeff noted that the total number of towers for each height were 'all multiples of two'. As Michelle and Milin explained to Jeff that the 'doubling pattern' was a result of the number of colours of cubes which were available, two in this case, Stephanie was silent.

During an earlier interview (see Episode 1), Stephanie also had noted the 'doubling pattern'. [9] Four days later during Episode 2, Stephanie confirmed that the number of towers had doubled each time the towers increased in height by one cube. She said that this

8 Milin indicated that there were only two possible towers 1-cube tall, one red and the other, blue. He then explained that since he was selecting from two color choices, he could add a red cube or a blue cube to the top of each tower 1-cube tall, to generate a total of four possible towers 2-cubes tall. Continuing this argument, Milin added a red cube and a blue cube to each of the four towers which were 2-cubes tall to create a total of eight towers 3-cubes tall.

9 By now, Stephanie had tried to build and to account for all possible towers that were 3, 4, 5, and 6 cubes tall.

'pattern' continued for taller towers such as those 10-cubes tall which she indicated that she 'figured out' was 1024.

Teacher:	Now you made towers of 5 in class and what did you get?
Stephanie:	Thirty-two.
Teacher:	Does it work the same way?
Stephanie:	Yeah.
Jeff:	They're all multiples of two.
Stephanie:	Yeah. The hard part is to make the patterns. Like you could… from now we know how to just like… Oh, you could give us a problem like how many in ten [towers ten cubes tall] and we could just go…
Teacher:	All right, how many in ten right… and you'd know the answer.
Stephanie:	Yeah. [She responded immediately] I know the answer. I figured it out… one thousand twenty-four.
Teacher:	One thousand twenty-four… are you sure?
Stephanie:	Uh hum.

Although Stephanie was convinced that the 'doubling pattern' which she discovered in Episode 1 would give her the total number of towers of a given height, she **repeated** her version of a **proof by cases** to **justify** her solution of eight possible towers 3-cubes tall. She remarked that finding the total number of towers of a particular height was easy, but could not envision the generation of the arrangements of cubes for each tower by using the 'doubling pattern'. Recall Stephanie's statement, 'the hard part is to make the patterns'.

Significance

During this small-group discussion, Stephanie recalled the 'doubling pattern'. She had the opportunity to listen to Michelle and Milin's attempts to explain how the pattern worked, but adhered to an organisation that displayed towers by cases, apparently satisfied that this made sense.

Episode 3: To what extent does Stephanie follow the 'tree' organisation?

To what extent, if at all, did Stephanie follow the reasoning of her three classmates about how the towers were generated from the 'doubling rule'? In order to pursue the question, Stephanie was asked to reflect upon and write about her 'method(s)' for finding and justifying all possible towers. In a subsequent interview, Stephanie was asked if she had thought further about her method(s). At this time she retrieved a wrinkled sheet of paper she had been carrying in her notebook (see *Figure 2*).

Stephanie's report indicated that she remembered her earliest strategy for finding towers by 'opposites', and recalled her argument by 'cases' as well as the 'doubling rule'. The interview that followed focussed on assessing whether Stephanie had insight into how the pattern was generated. When it was established that Stephanie did not appear to understand the generation of the pattern produced by the 'doubling rule', the focus switched to pursuing whether Stephanie followed the reasoning of Milin (Episode 2) in building towers. The interviewer read from Stephanie's paper, Interviewer: 'Finding these methods, I found a pattern for blocks of 1.' Okay let us talk about blocks of one. Can you show me the blocks of one, and how many?

In response, Stephanie reached for the blue and yellow cubes on the table and built two towers 1-cube tall, one blue and one yellow. The interviewer then began to record Stephanie's results by constructing a chart with a column to keep track of the 'number of blocks' in each tower and a column for the 'number of towers' for each height. Stephanie continued by building four towers 2-cubes tall in the following order: two yellows, two blues, blue on the bottom and yellow on top and yellow on the bottom and blue on top. The interviewer recorded with a tree diagram Stephanie's work as she built towers with plastic cubes. However, the interviewer recorded the towers in a different order than that which Stephanie produced.

Interviewer: What are the colours you started with?

Stephanie: Blue and yellow.

Interviewer: You started with blue and then you started with yellow, right? That is how you got the two. This is one high, right? [The interviewer wrote the B for blue and Y for yellow at the bottom of a sheet of paper. To the extreme left of the same line she wrote '1 high' and to the right she wrote '2'.] And then what did you do with the blue?

Stephanie: We added another one onto the blue for number two.

Interviewer: What colour?

Stephanie: Blue.

Interviewer: So you added a blue [She wrote a B above the first B and connected the two B's with a line.] Then what did you do?

Stephanie: Then we added a yellow to the yellow. [The interviewer wrote a Y above the first Y and connected the two Y's with a line.]

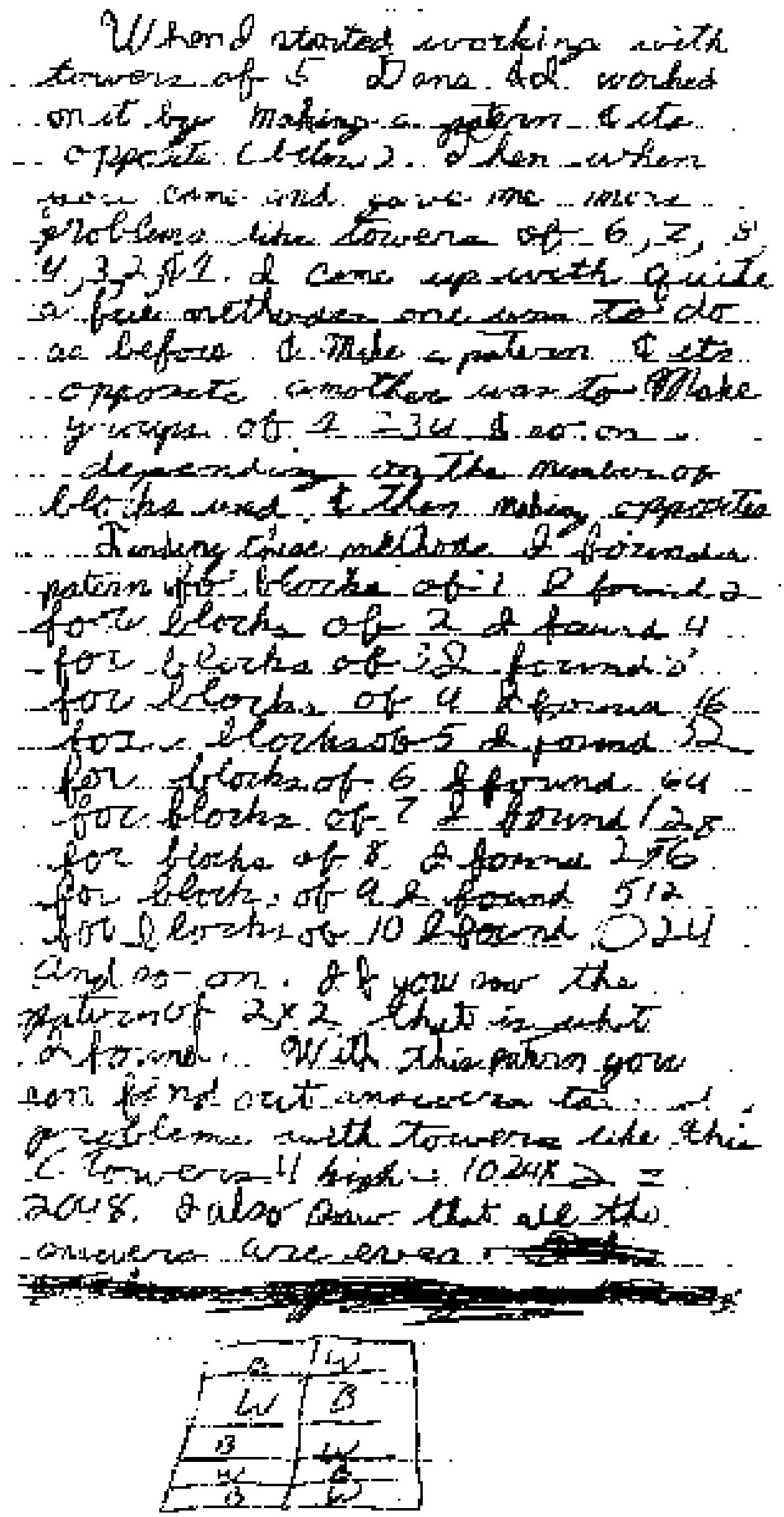
Stephanie then added the tower with blue on the bottom and yellow on top and the tower with yellow on the bottom and blue on the top to her set for towers 2-cubes tall. While recording the fourth entry for towers 2-cubes tall onto the tree diagram which was a blue cube on top of a yellow cube, the interviewer drew a line from the B to the Y. The result was a V shaped diagram over the blue tower that was 1-cube tall. The interviewer then said, 'This means on top of'. Stephanie replied, 'all right, whatever.' The interviewer had introduced the structure of the 'tree' to ascertain whether this method of generating towers would be recognised by Stephanie. However, Stephanie did not build in the same way.

Interviewer: Two high looks like this: blue on the bottom blue on top, blue on the bottom and yellow on top.

Stephanie: Yeah.

Interviewer: But you didn't do it that way. You did blue on the bottom blue on top and yellow on the bottom yellow on top.

Stephanie: Yeah.



When I started working with towers of 5 Dana & I worked on it by making a patern *(sic)* & its opposite (below). Then when you came and gave me more problems like towers of 6, 7, 8, 4, 3, 2, 1 I came up with quite a few methods one was to do as before & make a patern & its opposite another was to make groups of 1 2 3 4 & so on depending on the number of blocks used & then making opposites. Finding these methods I found a patern for blocks of 1 I found 2
for blocks of 2 I found 4
for blocks of 3 I found 8
for blocks of 4 I found 16
for blocks of 5 I found 32
for blocks of 6 I found 64
for blocks of 7 I found 128
for blocks of 8 I found 256
for blocks of 9 I found 512
for blocks of 10 I found 1024
and so on. If you saw the patern of 2 x 2 that is what I found. With this patern you can find out answers to this problem with towers like this towers 11 high = 1024 x 2 = 2048. I also saw that all the answers are even.

[The table referred to and drawn at the bottom of the page at left is:]

B	W
W	B
B	W
W	B
B	W

Figure 2: Stephanie writing about her methods for finding and justifying all possible towers.

Stephanie was asked to assume the role of teacher and to extend the 'tree' to towers 3-cubes tall and explain how the towers were being generated.

Interviewer:	Okay. Now don't do it, but tell me, what your reasoning is now. Cause you're the teacher now and I am the student now and this is the way that I imagine it…
Stephanie:	We're gonna build on from here.
Interviewer:	So this is blue-blue.
Stephanie:	Okay… we're gonna add another blue.

Stephanie continued to resist Milin's 'tree' method, that is, placing either a blue cube or a yellow cube to the top of each tower 2-cubes tall to produce all possible towers 3-cubes tall, and instead placed one colour cube on top of each of the four towers which were 2-cubes tall producing four towers 3-cubes tall. She then produced the absent towers 3-cubes tall by using trial and error.

The interviewer in reviewing Stephanie's eight possible towers 3-cubes tall again presented them in a way which suggested considering the two possibilities for each tower 2-cubes tall before considering any possibilities with another tower 2-cubes tall.

Interviewer:	So here is the [tower with] two blues on the bottom, right. So one thing you could do is put a blue [cube] on top.
Stephanie:	Uh hum.
Interviewer:	What is the other thing we could do?
Stephanie:	We could put a yellow [cube] on top.
Interviewer:	Is that right?
Stephanie:	Uh…um…
Interviewer:	Nothing else is possible, right?
Stephanie:	No, not with the two blues stuck together.

Stephanie then finished this method of placing a blue cube and a yellow cube on top of each tower 2-cubes tall to generate all towers 3-cubes tall.

Interviewer:	What if I have a yellow [cube on the top] and a blue [cube on the bottom]. What is possible? Don't do it. Just tell me.
Stephanie:	All right, uhm… you could put a yellow on the top or you could put a blue on the top.

When asked to report how many towers she had found which were 3-cubes tall, Stephanie replied eight. Challenged to consider how she had obtained eight, Stephanie replied, 'I didn't use my multiplication on that one cause I didn't know it [the 'doubling rule'] until after I got to like number five [towers 5-cubes tall]'. As she began to build taller towers, Stephanie continued to sometimes place a blue cube and then yellow cube on top of each tower of a particular height and on other occasions to retrieve towers in a less systematic way. She appeared to comply with the interviewer's method for generating towers reproducing 'chunks' of this organisation, but did not use this method consistently.

Significance

The interview revealed that Stephanie did not generate tower arrangements based upon the 'doubling rule'. Her methods for making the towers 'grow' in height did not suggest a systematic generation as she added to the top of towers. The interviewer then suggested an organisation by a tree diagram. She complied with the interviewer's suggestions for building up her towers in a more systematic way by modelling the interviewer's technique for towers of taller heights, but did not use this method which was introduced to her in a consistent manner.

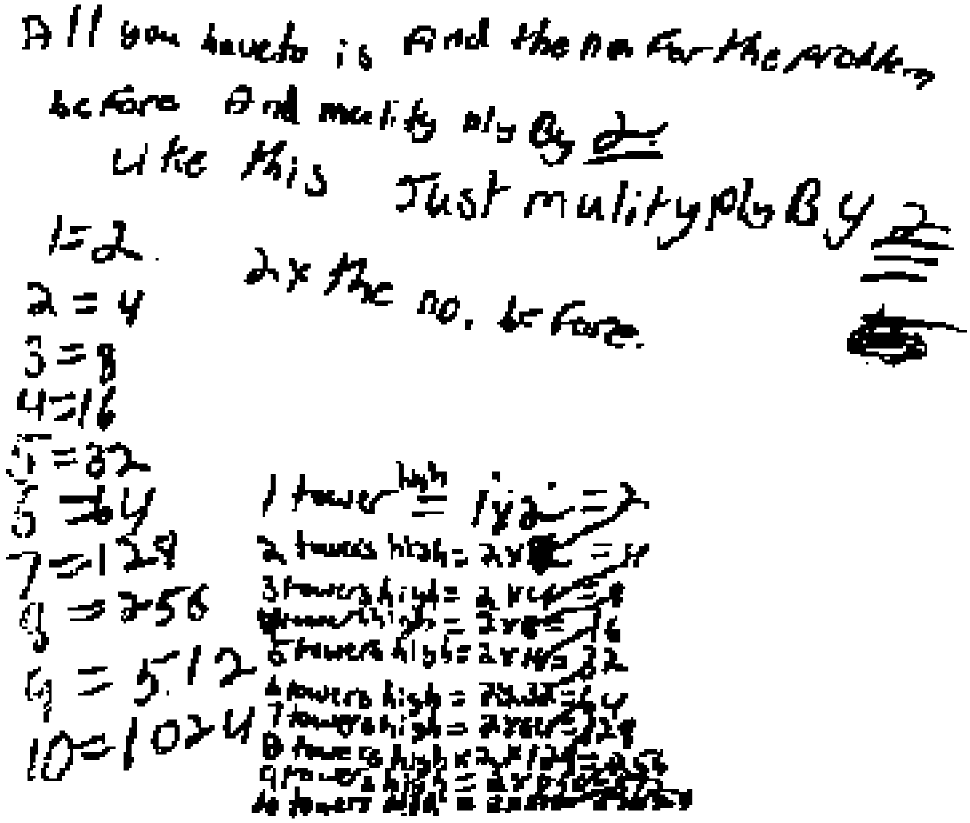


Figure 3: Stephanie's June 15, 1992 written assessment for finding all the towers of a given height.

Episodes 4 & 5: Stephanie used the 'doubling pattern' to check results during written assessments

Both with a partner in grade 4 (June 15, 1992), and individually in grade 5 (October 25, 1992), Stephanie was asked to write a letter to a student who was not present that would convince him/her that she had accounted for all possible towers 3-cubes tall without any duplication when selecting from two colours of cubes.

On the grade 4 written assessment, Stephanie quickly reproduced a version of proof by cases (for black and green cubes). She was then challenged to generalise her method by the teacher who asked: 'Suppose you want to find all the towers that are any number high?' Stephanie and her partner, Milin, produced the written explanation given in *Figure 3* above.

Stephanie then expressed renewed interest in the 'doubling' strategy,

> You know it's too bad we didn't know this [procedure] sooner… in all these problems. I'm trying to figure out with the blocks… then all of a sudden it popped into my head, 'Oh there's a pattern here'…

Milin then indicated that multiplying the total number of towers by two was directly related to the fact that there were two colours of cubes to select from.

> See. Look. There's only two colours… if there's three colours you times them [the total number of towers of a given height] by three.

In a grade 5 written assessment, Stephanie refined her version of proof by cases to justify her solution and used the 'doubling pattern' to monitor her justification (see *Figure 4*).

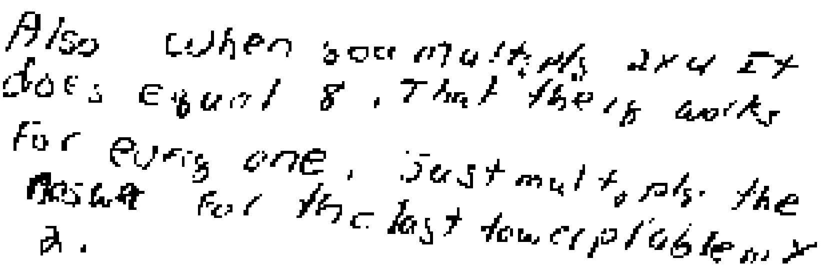


Figure 4: Stephanie's October 25, 1992 written assessment which uses the 'doubling rule' as a method of checking the total number of towers of a given height.

Significance

On two different written assessments — one, two months after Episode 3 and the other, five months later — Stephanie recalled the 'doubling pattern' and used it to check her solution by cases. These assessments suggest that Stephanie viewed the 'doubling pattern' as a quick means for obtaining the total number of towers of any height.

Episode 6: Guess my tower — Questioning the validity of the 'doubling method'

On February 26, 1992 in grade 5, Stephanie and her classmates had another opportunity to think further about the Tower Problem and its variations when presented with a new problem, Guess My Tower (see Note 2), which called for knowing all possible arrangements of towers 4-cubes tall when selecting from red and yellow cubes.

During this episode, Stephanie and Matt worked as partners. In trying to decide the total number of towers 4-cubes tall, Stephanie again referred to the 'doubling method' and predicted a total of sixteen towers 4-cubes tall. She made reference to her earlier procedure of doubling, and stated with conviction a rule for calculating the total number of towers of a given height based upon that pattern. However, she could not explain why this method worked.

Stephanie:	I don't know. It just works.
Teacher:	It just works?

Stephanie: It **always** works.

Stephanie and Matt continued to search for the sixteen towers, but could only account for nine that were 4-cubes tall when selecting from red and yellow cubes. Stephanie in conversation with the teacher expressed some discomfort with this discrepancy and reconsidered the validity of her 'theory'.

Stephanie: Well a couple of us figured out a theory because we used to see a pattern forming. If you multiply the last problem by two, you get the answer for the next problem. But you have to get all the answers. See, this didn't work out because we don't have all the answers here.

Matt: I thought we did.

Stephanie: No. I mean **all** the answers, **all** the answers we can get… I don't know what happened! Because I am **positive** it works. I have my papers at home that say it works. I know that you had to multiply it [the total number of towers of a given height] by something. Maybe it wasn't two because I know it worked. Maybe it was adding two…

Matt believed that there were only ten towers and challenged Stephanie's 'method'. In response, Stephanie tried to modify her theory to include only the towers whose heights were even numbers. She quickly abandoned this modification when she recollected her calculation for the number of towers that were 11-cubes tall:

Matt: See. This is all the possible answers for this [ten towers 4-cubes tall]. So that it doesn't always work.

Stephanie: I know. That's what's bugging me because I'm absolute… absolutely sure because we went … maybe it's only the even numbers or something… cause I went all the way up to [towers] eleven [cubes tall].

Matt: Yeah it's probably the even numbers. It's probably if it's an even number.

Stephanie: But I went all the way up to eleven with it. I had, like, one thousand… one hundred and twenty eight or something like that. I know because I went all the way into the thousands with this.

The teacher encouraged Stephanie to test her doubling idea and try simpler problems.

Teacher: Well… why don't you start with some of the smaller sized towers like two's or three's.

Stephanie: Yeah, because I know this works. I'm sure of it. We'll start with one and we'll multiply our way up.

Significance

The discrepancy between the number of tower arrangements that Stephanie found with her partner and the number of towers that she predicted from her 'doubling method' produced a conflict that led Stephanie to question and to modify her theory. However, the adjustments that she proposed still did not work. This discomfort may have encouraged Stephanie to pursue the teacher's suggestion to try her method on a 'simpler problem' (towers 2-cubes tall and 3-cubes tall). Stephanie now had the opportunity to explore more deeply, thus providing a window of opportunity to move beyond pattern recognition to the challenge to think about how the 'doubling pattern' worked.

Episode 7: Guess my tower — Regaining confidence in the 'doubling method'

As Stephanie and Matt continued to work on the 'Guess My Tower' problem it became evident that Stephanie was unable to recognise the generation of towers in the 'doubling pattern'. Matt, her partner, suggested that Stephanie's 'doubling' theory might be wrong, but Stephanie defended her 'method' and argued that they had 'goofed' in their attempts to identify patterns to generate all possible towers.

I think we goofed because I'm still sticking with my two thing [doubling the total number of towers]. I'm convinced that I goofed… that I messed up…

Significance

In Episode 7, Stephanie no longer attempted to modify her method for calculating the total number of towers, but instead chose to doubt her method for generating and organising all possible arrangements. Possibly all her prior experience with the 'doubling pattern' influenced her decision to defend **it** rather than the local organisations for tower arrangements which she had produced with Matt.

Episode 8: Guess my tower — Stephanie and Matt share their thinking with classmates

Later in the 'Guess My Tower' session, the teacher suggested that Stephanie and Matt discuss their ideas with some of the other children in the class. Stephanie and Matt circulated throughout the classroom and compared their results with other children. They visited Milin and Michelle who demonstrated a method for finding all possible towers using a 'tree' organisation.

The teacher suggested that Stephanie and Matt compare their solution of twelve possible towers 4-cubes tall with the solutions of other students. Stephanie and Matt joined Melissa and Bobby who had found a total of sixteen towers 4-cubes tall. Once this discrepancy in the total number of towers surfaced, Stephanie explained to the teacher, Bobby and Melissa how her 'doubling method' would give her a total of sixteen towers, but that she and Matt were only able to find twelve arrangements.

When we multiplied it out [by two] we got sixteen possible towers]… but we weren't able to find that many tower arrangements]. We were only able to find… like twelve.

Significance

What was evident from this episode was Stephanie's growing awareness and discomfort regarding the discrepancy between her 'doubling method' and the number of tower arrangements that she and Matt found. Although she had used the 'doubling pattern' to determine the total number of towers, she was not able to build the total number of predicted arrangements.

Episode 9: Guess my tower — Stephanie's initial attempt to connect her 'doubling method' to the generation of tower arrangements

At Stephanie's expression of her inability to reconcile the 'doubling method' with generating all possible towers, the teacher suggested that Stephanie try to explain how the towers 'grow' in height to Matt, Melissa and Bobby. Stephanie attempted to respond to the teacher's question of how towers 2-cubes tall could be built from towers 1-cube tall. She began by using red and yellow cubes to build the two towers 1-cube tall and the four towers 2-cubes tall and placed these towers into two sets.

There are the parents [the red and yellow towers 1-cube tall] and the parents have children [the four towers 2-cubes tall], okay? They have four children… and their children have kids [eight towers 3-cubes tall]… you start out with two [towers 1-cube tall], okay? You're convinced, right? You're convinced this is really two? Then you have to move to the next one, okay? Now for towers of two, there's only four. Are you convinced that there's only four?

The teacher challenged Stephanie to explain how she got the four towers 2-cubes tall from the two towers 1-cube tall. Stephanie responded to the group by considering the completeness of her justification for each individual case rather than responding to the possibility that the cases might be in some way related.

Once there's no more, there's absolutely, positively no more… you can't build any more with one [cube]. So you build the next number and that number is two, okay? So you have four [towers] of [height] two.

The teacher indicated that this 'leap' from two towers 1-cube tall to four towers 2-cubes tall was too fast for her to follow, and she didn't understand how Stephanie was getting the towers of height two from the towers of height one.

Significance

The episode demonstrated Stephanie's willingness to consider the completeness of individual sets of towers for a specified height [all possible towers 1-cube tall or all possible towers 2-cubes tall]. She did not demonstrate how one would generate all possible towers from one height to the next.

Episode 10: Guess my tower — Matt uses the 'tree' method presented earlier by Michelle and Milin

Matt responded to the teacher's query about how the towers 'grow' and demonstrated to Stephanie, Bobby and Melissa that an existing tower could be built one cube taller by adding either a red or yellow cube, pointing out that there were no other possibilities because there were only two colours.

So you have the same red [cube] on the bottom [two towers with a red cube in the bottom position]. You add a red [cube] or a yellow [cube] on top [of the red cube on the bottom]. You have the same yellow [cube] on the bottom… [two towers with a yellow cube in the bottom position] but you add a red [cube] or a yellow [cube] on the top.

Significance

Stephanie appeared to be listening very carefully to the justification presented by Matt. Although she had made earlier reference to a 'family tree' and the idea of generations [parents and children], she had not been able to move beyond her recognition of a pattern to form the basis for a method of justification related to that 'pattern'. It was possibly at this time that Stephanie connected the 'pattern' or 'method' to a reason for why it worked.

Episode 11: Guess my tower — Stephanie's 'Aha'

The teacher asked Stephanie if she could explain how Matt was making the towers 'grow' in height. Stephanie's face expressed excitement as she indicated that each tower was built up by adding either a red cube or a yellow cube to the top of it to produce two new ones of the next height. This procedure for building matched her 'doubling' idea.

Stephanie: Each one [tower 3-cubes tall] reproduced. It gives like it… it had like two [towers 4-cubes tall].

Teacher: So this one [tower] has two?

Matt: Each one reproduced.

Teacher: Two babies?

Stephanie: Yeah.

Stephanie and Matt then constructed a 'tree' with the actual towers which accounted for all possible towers of heights 1, 2, 3, 4-cubes tall. The teacher asked them to recall their initial prediction for towers 4-cubes tall using Stephanie's 'method'. They recalled their prediction of sixteen towers and counted the arrangements that they had just built. To Stephanie's delight this new method of 'building up' with red and yellow cubes matched her 'doubling theory'. She showed satisfaction that she 'was right all along' and happy 'that her rule worked'.

Teacher: How many do you predict?

Stephanie: Sixteen [towers 4-cubes tall].

Matt: 2, 4, 6…

Stephanie: 2, 4, 6, 8…

Together: 10, 12, 14, 16!

Stephanie: [exuberantly] Yes! I knew it! I knew it! I knew it!

Matt: I knew it? I knew it? I knew it?

Stephanie: I told him all along. I was right… so we're going to talk to the class about this?

Teacher: I think so.

Stephanie: Want **me** to do it?

Teacher: Do you [to Bobby and Melissa] understand what they did?

Stephanie: I understand. I'm just very happy that my [doubling] rule worked.

Teacher: Your rule worked. But… you know what I think is valuable for people to understand is to know why that rule works.

Stephanie: Well I know what it is now I figured it out! I'm just happy that it worked.

Significance

This was the first documented instance of Stephanie's recognition of the relationship between the 'doubling pattern' which she had noted almost one year prior to this session and the reason why it worked.

Episode 12: Stephanie shared her understanding of the 'doubling method' with her classmates.

Stephanie, with great enthusiasm, presented the 'tree' justification for all possible towers of heights 1, 2, 3, and 4-cubes tall to the class (see *Figure 5*).

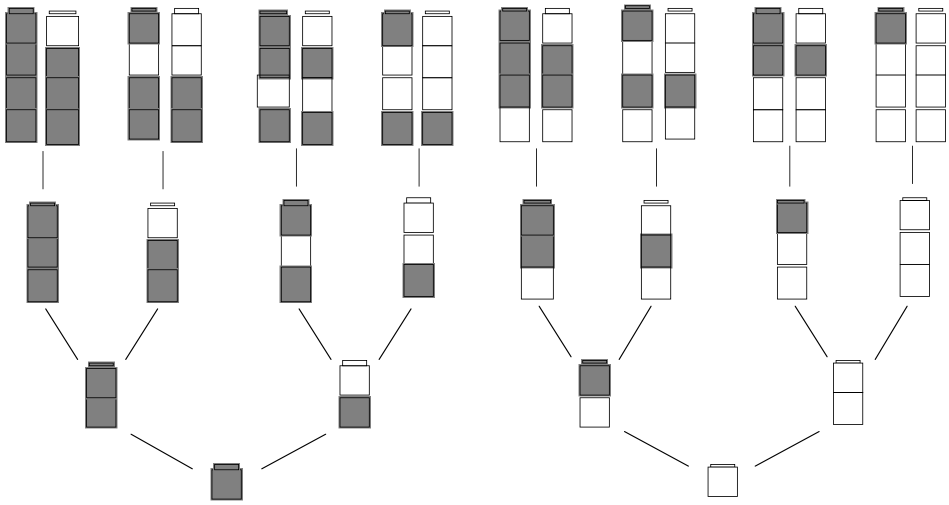


Figure 5: A replica of Stephanie's 'tree model' which she built with plastic cubes to justify her solution on February 26, 1993.

Stephanie: One red, okay? [She referred to a red tower 1-cube tall.] And I have a yellow [She referred to a yellow tower 1-cube tall.] and for each of these you can make two [towers 2-cubes tall] because all you have to do is… you can add on a red [cube] to the red [tower 1-cube tall] and a yellow [cube] to the red [tower 1-cube tall]… and for the yellow [tower 1-cube tall] you can add on a red to the yellow and a yellow to the yellow, [producing a total of four towers 2-cubes tall] okay?

Michelle: So you don't have to look for duplicates.

Stephanie then continued by presenting how the four towers 2-cubes tall became eight towers 3-cubes tall.

> Then each one of these [four towers 2-cubes tall] has two, okay? [Each tower 2-cubes tall can be built up into two possible towers 3-cubes tall.] If this is like a family tree… the mother [She points to one tower 2-cubes tall] the parents [She points to the four towers two cubes tall]… and then six kids

> [points to the eight towers 3-cubes tall]… well no, actually eight kids… then if they [each tower 2-cubes tall] have eight kids and each one of them [each tower 3-cubes tall] has two kids [two towers 4-cubes tall]. You can add one red, one yellow [to the top of the first tower which is 3-cubes tall] one yellow, one red [to the top of the second tower 3-cubes tall], one red, one yellow… cause each one of them is different … you keep adding on. And then here [points to the 16 towers 4-cubes tall] you can add the exact same pattern.

Significance

Stephanie presented this method of justification with a great deal of confidence to her classmates. It was clear from this episode in her video portfolio that she had taken 'ownership' of this method of justification. As she presented this organisation, she did not attribute it to the work of Michelle, Milin or Matt. Instead she presented it independently because it **now** made sense to her.

Conclusions and implications

Stephanie did not learn by being shown or by being told what to do. However, she ultimately **did** work out a solution when she became 'cognitively' involved and when the need to reinvent[10] arose, that is, when it became meaningful for her to learn the idea.

For Stephanie, the rule became a useful tool for monitoring the results of her method for keeping track of all possible towers that were organised according to cases. Beginning with the recognition of a pattern to the discovery of a 'doubling rule', Stephanie had little, if any, interest in understanding how the rule worked, although she had several opportunities to hear it explained by others. On one occasion, during the Episode 3 interview, Stephanie was actually **shown** how two new towers, one cube taller, could be generated from an existing tower by placing one colour cube and then the other colour cube on top of it. She then imitated the 'procedure' and gave an 'explanation' for why it worked. Still again we see later that these actions did not guarantee that she understood what was happening. How can this be explained?

In the instances when the rule was explained to Stephanie, she was inactive in the observed constructions. Her understanding of why the doubling pattern worked was not integrated into her schemata until Episode 12 when it became *her* invention. The opportunity arose when it became **important to her** that the method made sense and that the discrepancy between earlier work and the application of the rule arose. It was then that **she chose** to become mentally engaged. Documentation for these events came from Stephanie's writing, from in-class explanations, and from responses in follow-up interviews.

What other conditions might help explain Stephanie's conceptual change from pattern recognition to theory posing about how the rule worked?

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10 We mean reinvention in the sense described by Freudenthal (1991). For a detailed discussion, see particularly Chapter 2.

The classroom environment was crucial. Children were provided with sufficient time for exploration and reinvention. Stephanie worked in classroom sessions that were often extended beyond the norm and in classroom organisations that encouraged students to work on a problem alone or with others for hours when necessary.

Freudenthal (1991) wrote:

> Learners should be allowed to find their own levels and explore the paths leading there with as much and as little guidance as each particular case requires. (p.47)

He also wrote:

> Guiding means striking a delicate balance between the force of teaching and the freedom of learning. (p.55)

If we agree with this position (and we do), the role of the teacher is central. First, the teacher must be aware of the distinction between student *imitation* and student *reinvention* and of the differences in students' thinking. She must be ready to seize the opportunity to provoke thought and guide reinvention.

With respect to reinvention, Freudenthal (1991) raised the question: 'where to?'. He answered: 'to an activity', and then continued:

> In other words, the learner should reinvent mathematising rather than mathematics; abstracting rather than abstractions; schematising rather than schemes; formalising rather than formulas; algorithmising rather than algorithms; verbalising rather than language — (p.49)

Notes

1. We would like to acknowledge the efforts of the following graduate student researchers: Mary DeHart, Siham Alfred, Dawn Guarneri, Kerith Glickman, Melissa Lee, Elise Pattison and Jill Tomaini.

2. The problem known as 'Guess My Tower' was given to students in the following form.

   This game is played by choosing one of four possibilities for winning and then picking a tower out of a covered box. If the tower you pick, matches your choice, you win the game. The box contains all possible towers that are three cubes tall that can be built when you select from cubes of two colours, red and yellow. You are given the following possibilities for a winning tower:

   1. *All cubes are exactly the same colour;*
   2. *There is only one red cube;*
   3. *Exactly two cubes are red;*
   4. *At least two cubes are yellow.*

   Which choice would you make and why would this choice be better than any of the others?

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