MEM/USU
— 1998 —

Série Reflexões em Educação Matemática

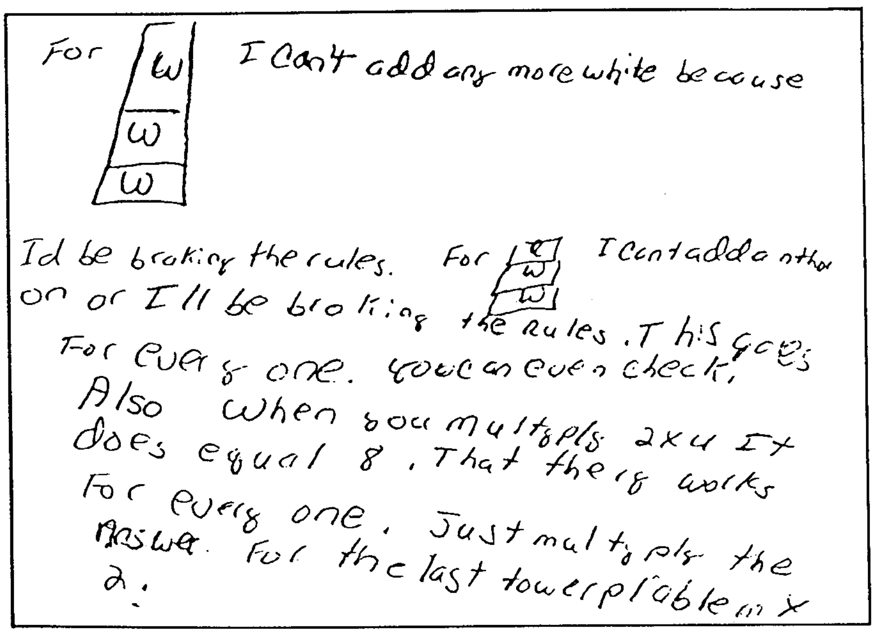
ISSN 0104-9720

UNIVERSIDADE SANTA ÚRSULA

MESTRADO EM EDUCAÇÃO MATEMÁTICA

Can teachers help Children make convincing arguments?

A glimpse into the process



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volume 5

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Chapter Two:
The Nature of Learning

A model for the development of mathematics teachers was proposed and implemented in a K-8 school district in order to bring substantial changes in long established views and practices of teachers (Maher, 1998). The long-term collaboration with the Harding School, in Kenilworth, New Jersey was developed in 1984 and continues today. It was at the Harding School in which the longitudinal study of the development of mathematical ideas in children began with an initial group of children, identified in grade one, and with whom we continue to work as they attend nearby high schools.

The Rutgers-Kenilworth partnership became a clinical investigation of the validity of certain hypotheses of learning, and their application for effective in-service teacher development. The original model for the development of teachers "sought to provide a set of interrelated experiences for teachers to enable them to develop a perspective on mathematics instruction as the creation of a classroom environment in which children construct the concepts and ideas in a problem-solving context" (Maher, 1988, p. 299).

It called for studying, by careful observation and analysis, the mathematical thinking of children who were actively engaged in doing mathematics. The context for that learning required important considerations that were described early in the project:

> In simplest form, it sees learning as arising in a problem-solving context in which students are engaged in investigations in mathematics that give them an opportunity to explore patterns, make conjectures about their character, test hypotheses for their effectiveness in problem solving, and reflect on the formulation of the concept for use in analogous problem-presenting situations. In a constructivist context, important considerations are: (1) motivation as arising from the need

to solve a problem, (2) with the aid of physical embodiments, living through the experience of learning mathematical concepts, operations, patterns and structures, and how they are connected, (3) the extension and development of concepts and operations through generalization in multiple representations of them, (4) the application of the generalization to similar problems, and (5) provision for individual differences in children in the course of problem solving.

(Maher, 1988, p. 297)

Teacher as Learner of Mathematics

The model that guided the long-term intervention in the Kenilworth schools called for teachers to build a broad understanding of the mathematics they were expected to teach, and as their own understanding deepened, "to reflect on the processes by which this knowledge was acquired" (Maher, 1988, p. 300). In a similar way, an emphasis on justification and proof making in schools starts with a parallel emphasis in the proof-making mathematical knowledge of teachers who, like their students, can profit from explorations that evoke model building. Courses and workshops give opportunity for teachers, as learners, to think more deeply about the tasks they are challenging their students to investigate. Teachers, in reflecting on their own solutions and observing the solutions of colleagues, witness and give thought to the variety of representations that are built, as well as to the processes involved in shaping and extending their ideas.

As an example, consider the *Building Towers* problem.[8] Many teacher learners begin, as do young children, by randomly buiding towers. Soon relationships such as pairs of "opposites", that is, opposite colored cubes in corresponding positions, are identified. Checking, in the early building,

[8] *Brandon's work on towers and pizzas provides another, detailed, example, in Chapter 6. Also, see Davis & Maher, 1993; Maher & Martino, 1997; and Maher & Martino, 1996a.*

is based on recognition of duplicates. Patterns and relationships between towers generate new sets and opposites of those sets. Checking becomes more sophisticated as students begin to monitor for duplicate arrangements by comparing a newly generated tower to ones that have already been built. This leads to other local organizations among sets of towers and ways of generating new towers from existing ones such as "inverting" a tower and its opposite to form another collection. What often follows is a recognition that the new schemes still are inadequate to account for all possible towers. Frequently a search for other schemes leads to a recognition of conflicts between different organizations in which duplicates appeared. Focusing on subsets of these schemes, such as exactly all of a color or one of a color supports the movement to a more global organization.

We have observed that, through investigation and free play, teachers develop organizations that lead to building representations, revising and / or modifying them, sometimes discarding them, and cycling through this process. In so doing, they develop arguments to justify their ideas. During the problem solving explorations, new ideas appear and form the basis and motivation for the development of others. Challenged further by problem extensions, teachers can revisit ideas that, in earlier mathematics learning, were only partially, if at all, understood. By entering new knowledge representations of the input data, deeper understanding can evolve and extensions to other ideas can be constructed. A joy of teaching is to witness the delight of teacher learners who recognize the connectedness of mathematical ideas, such as with the binomial expansion and Pascal's triangle in the *Building Towers* extensions. The notations invented by teachers to express their new ideas pave the way to understanding other symbol systems that were previously meaningless and inaccessible.

Teachers have reported that engagement in these investigations has helped give meaning to the mathematical ideas, connect them to others, and extended them in powerful ways. The elegant and sophisticated constructions produced by students and teachers suggest the appropriateness of these and other tasks for building unification of ideas and concepts, hitherto viewed as compartmentalized and discrete. In fact, very substantial changes have been documented by studies[9] about teaching and children's development of mathematical ideas.

Teacher as Learner of Children's Thinking

After working on the *Building Towers* problem, teachers report that it is helpful to study the varieties of ways that children have thought about the mathematical ideas that were encountered when working on the task. They indicate that a careful study of children's thinking helps them to recognize the variety of diverse representations developed by children. One way to introduce teachers to children's thinking about justification is to view videotapes of children doing mathematics and to study, using transcripts of the tapes and children's accompanying work, the development of children's ideas.

Two tapes that have been successfully used in courses and teacher workshops about proof making in children are *The Gang of Four* and *Brandon and the Pizza Problem*. These are briefly described below. A detailed analysis of the Brandon story is given in Chapter Six.

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[9] *See Landis & Maher, 1989 and the following Rutgers dissertations about teaching: Landis, 1990; O'Brien, 1995; Schorr, 1996; and Thatcher, 1995. See also Martino, 1992.*

The Gang of Four, Grade 4
(March 10, 1992)

A videotape shows four nine-year old children, Stephanie, Jeff, Michelle and Milin, seated in a conference-like setting, engaged in a discussion about the *Building Towers* task. In this tape Stephanie tries to convince Jeff that she has found all possible towers 3-tall that could be built, selecting from plastic cubes in two colors, blue and red. In so doing, she swiftly and confidently presents a justification for her solution by organizing the tower combinations into five cases (no blue, one blue, two blue "stuck together", three blue, and two blue separated by one red.) The other children, Michelle and Milin, join Jeff and Stephanie in a very lively and thoughtful discussion. In the same tape, Milin presents an inductive method for organizing the towers.

Notice Stephanie does not use plastic cubes to represent her idea, but rather has invented a notation using letters arranged to represent the various towers. (See Figure 1.)

Individual Written Assessments
(Octuber 25, 1992)

Seven months later, a written version of the *Building Towers Problem* was given to the children. The durability, refinement, and extension of the children's ideas can been seen from their written accounts.

Stephanie. In Figures 2a and 2b, we see a refinement in Stephanie's March 10th representation. She begins by referring to towers 2-tall and writes: "All you have to do is multiply 2 x the number you would get for towers of two. So it is 2 x 4: I will prove it." Notice the care with which she indicates her organization, indicating that she will be classifying according to "color order". She uses the same care to introduce her coding: "R stands for red & W stands for white". The arrangement that is now displayed indicates that the five categories, insisted on in the March 10th discussion, have been collapsed into four discrete categories (See Figure 2a.). Above each tower representation, she specifies the classification. Stephanie, (See Figure 2b.) then switches to a discussion about the uniqueness of a particular tower. She displays a drawing of a tower with each white cube labeled with the letter W, to indicate a white cube. She proposes that there could only be one 3-tall tower with all white cubes "because, I'd be braking [sic] the rules", that is, she would be contradicting the problem requirement of the given tower height of 3-tall. Stephanie's written work displays not only the durability of her earlier ideas, but also their extension and refinement as new input data, arising out of conversation with her classmates, is entered into her original representation. [10]

Milin. Milin's[11] generalization of a doubling rule is indicated by noting that there would be two towers, 1-tall; 4 towers, 2-tall; 8 towers, 3-tall; and 16 towers, 4-tall (See Figure 3a). He displays drawings of 1-tall and 2-tall towers and labels the blocks using R for red and W for white (See Figure 3b). He produces the two towers predicted by the first doubling, and then the four towers, by the second iteration.

Jeff. Jeff's written work shows a different method for generating all possible 3-tall towers. In his letter to Ronald McDonald, Jeff specifies the notation used to label each tower in the collection. He writes: "W stands for white and M stands for maron [sic]." He, then, displays a drawing of eight towers with each cube labeled using an M or a W. The first tower is all maroon, followed by a tower of all white. Reading from bottom to top, he displays the next three towers with exactly one maroon; reading from top to bottom, the last three towers contain exactly one white. Jeff writes: "I moved maroon up untill [sic] it got to the top then I did the same thing but I used white. The other 2 are all white and all maroon." (See Figure 4).

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[10] *For further discussion of the development of Stephanie's ideas, see Maher & Martino, 1997, Maher & Martino, 1996a and 1996b.*

[11] *See Alston & Maher, 1993, for a more detailed analysis of Milin's development of proof by induction.*

Brandon and the Pizza Problem

Brandon, a nine-year old boy, worked in his classroom on both the *Building Towers* (November, 1992) and *Pizza Problem* (March, 1993) tasks. A study of the videotape data enables us to observe his construction of an initial representation of a solution to the tower problem and, later, the pizza problem. In a classroom written assessment about the tower problem (December, 1992) and in a follow-up interview (April, 1993), the consistency and stability of Brandon's original constructions are apparent.

Initially in the November, 1992 exploration, Brandon solved the 4-tall tower problem by showing that he found sixteen towers by identifying a tower and its opposite. He said he thought he had accounted for all of them because he could not think of any more. Four months later, Brandon worked on *The Pizza Problem.* He invented a notation of 0's and 1's to keep track of his topping choices and recorded his findings on a table that he built. Brandon's organization enabled him to account, by cases, for all possibilities.

During the April 1993 interview Brandon was asked about his solution to *The Pizza Problem.* When he completed his explanation, he was asked if the pizza problem reminded him of any other problem he had done. Brandon recalled the towers task and immediately sought to find all possible 4-tall towers that coud be built when selecting from two colors. In this second construction, he again found sixteen towers. Again, he organized the collection by eight sets of opposites. He then spontaneously noticed that pairs of opposite towers could be reorganized in such a way as to map into his coding scheme for pizzas.

The videotape interview enables us to observe Brandon explaining to the interviewer his representations to both problems. In so doing, Brandon is able to reflect upon and modify his original problem solving for the tower problem. Brandon, who originally had constructed two <u>different</u>

representations, one for the tower problem and the other for the pizza problem, spontaneously recognized a relationship which enabled him to coordinate them into a single representation. He established an isomorphism between the towers and pizzas, and physically mapped each tower onto the appropriate row of his table of zeros and ones. Brandon, thoughtfully engaged in his problem solving, coordinated the two representations and constructed a more elegant and powerful representation for towers. Brandon's new actions on the towers, driven by his newly constructed representation for pizzas, enabled him to rebuild a representation of the input data for towers. In cycling through this process, Brandon was able to build a convincing argument, according to cases, for both the pizza and tower problems.

Building a personal philosophic perspective

Analyzing the mathematical thinking of children by examining videotape records of problem-solving sessions helps teachers think more deeply about children's learning and the complexity of the development of ideas (Davis, Maher, & Martino 1992). Accompanied by efforts to build a deeper understanding of mathematical ideas, teachers' guided reflection on their own and children's learning will, it is expected, help to develop a personal, philosophic perspective on the learning and teaching of mathematics.

THE GANG OF FOUR

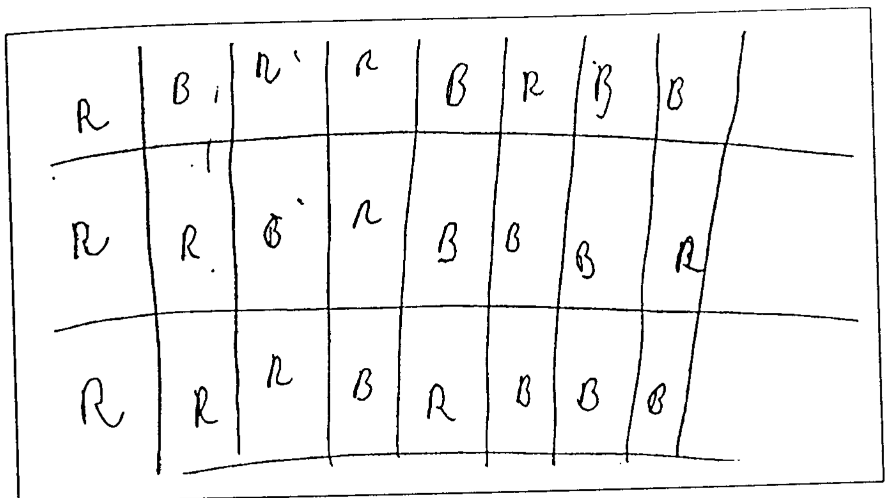


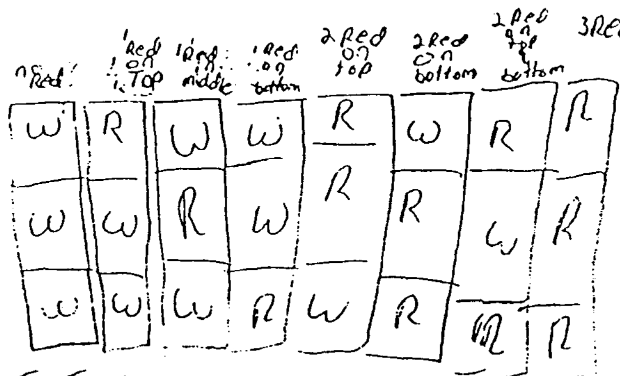
FIG. 1

Name Stephanie Date

Please send a letter to a student who is ill aand unable to come to school. Describe all of the different towers that yyou have built that are three cubes tall, when you had two colors availablle to work with. Why were you sure that you had made every possible tower and had not left any out?

Dear Laura,

Today we made towers 3 high and with 2 Colors. We have to be sure to make every posible pateren. There are 8 patterns total. I know because all you have to do is multyply 2 X the number you would get for towers of two. so It is 2 X 4: I will prove it. IF I put the towers in Color order The colors are red &whit. R stands For Red & W stands For white.



IF this doesent Convince you I tell you more → over →

FIG. 2a (Stephanie)

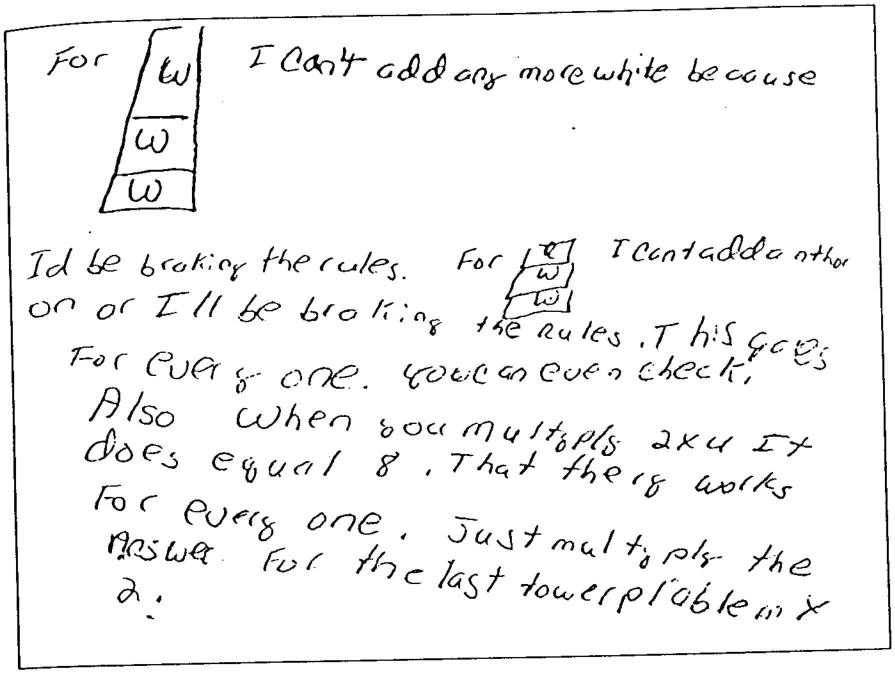


FIG. 2b (Stephanie)

Name *Milin Patel* Date____________

Please send a letter to a student who is ill and unable to come to school. Describe all of the different towers that you have built that are three cubes tall, when you had two colors available to work with. Why were you sure that you had made every possible tower and had not left any out?

?

To a Person,

I knew hows many their are because 1=2 2=4 3=8 4=16 5=32 and on the bottom is all of the ways.

R W B

2
4
8
16
32

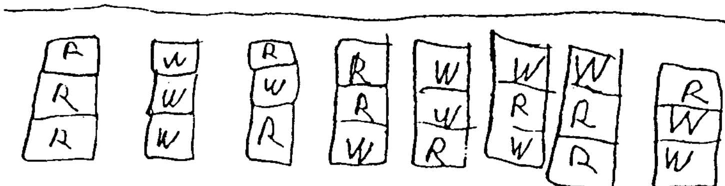


FIG. 3a (Milin Patel)

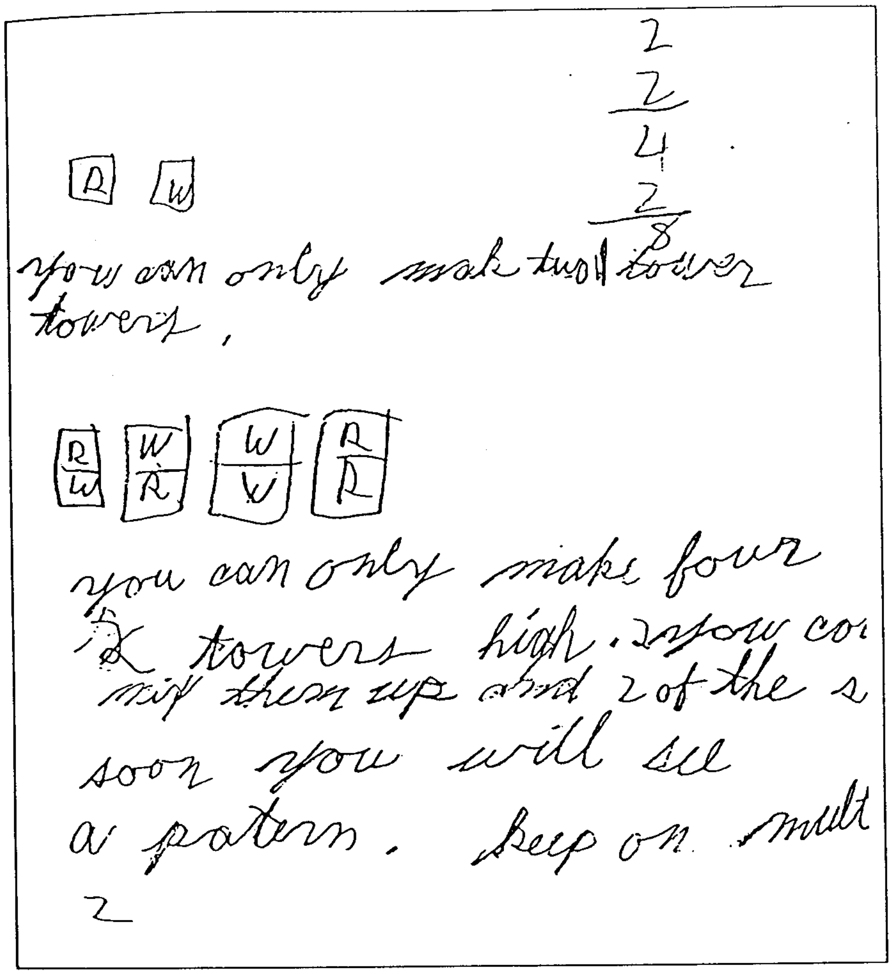
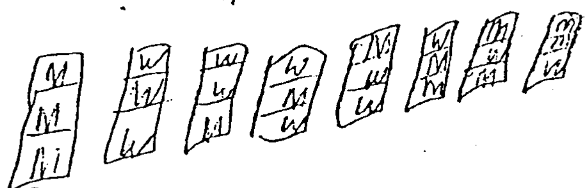


FIG. 3b (Milin Patel)

Name Jeff Hoal Date

Please send a letter to a student who is ill and unable to come to school. Describe all of the different towers that you have built that are three cubes tall, when you had two colors available to work with. Why were you sure that you had made every possible tower and had not left any out?

Dear Ronald Mcdonald,
You missed a great Day of math. Amy came and we worked with unfix cubes. We had to make towers with 3 cubes. Here's how they looked W stands for white and M. stands for Maron



I used the step method. I moved maroon up. untill it got to the top then I did the same thing but I used white. The other 2 are all white and all maroon.

FIG. 4 (Jeff)