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Using Videotapes to Study the Construction of Mathematical Knowledge by Individual Children Working in Groups

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Videotaping small groups of students in a regular classroom environment makes it possible to study individual student cognitive growth in a social setting. The present report deals with student development of some new mathematical ideas over an extended period of time.

KEY WORDS: Mathematics; representation; videotape; elementary.

INTRODUCTION

Videotape may do for mathematics education what the microscope did for biology—it may allow us to see many things that would otherwise remain invisible. By recording (on videotape) the mathematical work of students, following these same students for several years, and analyzing these tapes carefully, it is possible to observe the development of mathematical ideas over time.[3] When—as in the present report—this recording is done in a regular classroom setting, one is able to study individual student cognitive growth in a social setting, and thus gain insight into how social processes influence personal cognitive development.

In the case reported here, a combinatorics task was presented to some second graders. This same task was presented to them again five months later, when they were third graders. We give some data on their behavior in each case and relate their performance to what is known about instruction that took place in the classroom.

The Students

We report on six children, three girls (Dana, Jaime, and Stephanie) and three boys (Brian, Jeff, and Michael). In both second and third grades, mathematics instruction made use of small groups and was based on an expectation of considerable student initiative in devising ways to solve mathematical problems. In the second grade, the students worked in triads; at one table, Dana (D), Stephanie (S), and Michael (M) worked together (we call this group I); at another table, Brian (B) and Jeff (J) worked with Jaime (Ja) (group II). In the third grade, the children worked in pairs. What we will call group III consisted of Dana and Stephanie; group IV, of Jaime and Michael; and group V consisted of Jeffrey and Brian.

The Mathematical Task

The task given to the children was a word problem, for which they were *not* told in advance any method for solution. The problem went as follows:

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[3]The videotaping and analysis reported here is part of a six-year study of eight children, which is now in its fourth year. The teachers of these children are participants in a long-term teacher development project (Maher *et al.*, 1991). The children attend a K-8 elementary school that serves a working-class neighborhood.

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> Stephen has a white shirt, a blue shirt, and a yellow shirt. He has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

Data Collection

This word problem was given to the entire class, which was organized to encourage individuals and groups to work at their own pace and without teacher intervention. Following the group working sessions, the children were asked to share their group ideas with the entire class. After the sharing session, children were individually interviewed about the problem task. Data for this study came from children's written work and analyses of videotape transcripts from the following three sources: (1) classroom videotaping of the two triad groups (for grade two) and the three pair groups (for grade three); (2) classroom videotaping of each group sharing their solution strategy with the rest of the class; and (3) videotaping of interviews with the individual children following each class session.

THEORETICAL BASIS FOR THIS STUDY

Every study must begin with some theoretical framework, whether explicit or implicit, whether deliberate or unexamined. The conceptual basis for this study is that thinking about a mathematical situation involves cycling through a series of steps (Davis, 1984). First, it involves building a representation of the input data. From this data representation, memory searches are carried out to construct a representation of relevant knowledge that can be used in solving or trying to solve the problem.

A mapping between the data representation and the knowledge representation is constructed, checked out, and if judged to be satisfactory, is used to solve the problem. Moving from a data representation to a knowledge representation that might be useful in solving the problem may not be automatic. As the learner attempts to map the data and knowledge representations, checks are made along the way and other knowledge may be entered. In the process, some representations are rejected and/or modified. The learner cycles through this entire sequence, or parts of it, many times before constructing a "final" method of dealing with the problem. When the constructions appear satisfactory, other techniques associated with the knowledge representation may be applied to carry out the solution to the problem. Examples illustrating how the data representation of the problem statement are gradually constructed by individual students are given in Davis and Maher (1990).

What will emerge as of special interest in the present study are the relatively "primitive" foundation blocks from which these representations can be constructed. The process of building up representations from cognitive building blocks is referred to as "assembly" (Davis, 1984). In general, it is argued that effective building blocks are, at the beginning, the result of *experience*, although this is not always "concrete" experience (but it very often is). Of course, as the use of these blocks develops in more abstract or more generalized situations, more "abstract" building blocks come to be synthesized, first in a metaphoric way, and later in the form of truly abstract characterizations.

One can illustrate this development by considering the case of the concept of "function." In its earliest versions, the idea of "function" is merely a matter of being able to give certain answers and recognizing that these answers are dependent upon something else (as one might be able to give the price for a purchase of pencils, at ten cents apiece, if one knew how many pencils are to be bought). In the course of this work, one begins to use—perhaps even to invent!—some method for writing what mathematicians call *variables.* With more experience, one comes to recognize some of the essential features of tables, graphs, formulas, etc., as representations of functions, and one begins to see what all of these separate instances have in common. "Functions" come to be *things* that one can deal with, and one learns more and more ways of dealing with them. At some later stage, one can reformulate this into something akin to the abstract definition in terms of a set of ordered pairs (and so on).

As this development progresses—and usually *before* one gets to the "ordered pairs" abstract definition—one is likely to draw on a metaphoric use of earlier concrete experience, perhaps thinking in terms of *mappings* or even in terms of pieces of yarn extending from each possible "input" to the corresponding "output." Of course, even the words "input" and "output" themselves represent this use of metaphor, in terms of "putting something in" and "getting something out"—remnants of earlier concrete experience with some kind of gadget or apparatus or arrangement, perhaps like a machine where you toss oranges into a hopper and get orange juice out of a spigot.

Seymour Papert (1980) refers to this process of building abstract ideas on a metaphoric use of previous (possibly concrete) experience when he describes how important for later learning was his early opportunity to play with gears and to observe the way that turning one gear would result in the turning of another, perhaps faster, perhaps more slowly, perhaps in the opposite direction: "I became adept at turning wheels in my head and at making chains of cause and effect . . . . I believe that working with differentials did more for my mathematical development than anything I was taught in elementary school. Gears, serving as models, carried many otherwise abstract ideas into my head." (Papert, 1980, p. vi). We would phrase this differently: Gears, serving as cognitive building blocks, made it possible for Seymour Papert to build mental representations for other ideas that might seem unrelated—and unrelatable—to gears, ideas such as cause and effect, ratio, and so on. This pattern of building upon earlier experience is of particular significance in the present study, because we shall see some children building upon previous experience—using previous experience as *assimilation paradigms* or "internal metaphors"—in situations where most adults expect the children to operate on a far more abstract level. Metaphoric thinking and true abstraction are two very different ways of dealing with mathematics, as a growing collection of instances is making clear. It is important for teachers to be familiar with both types of thinking and to see how one type builds on the other.

GRADE TWO: THE SHIRTS AND PANTS ACTIVITY

The activity in grade two will be reported separately for the two groups of children: Group I consisting of Dana (D), Stephanie (S), and Michael (M) and group II consisting of Jaime (Ja), Jeff (J), and Brian (B). It is important to notice that, although the children did work in groups, their individual ways of representing the problem, and the methods for solution which they invented, were their own, and usually different from those of others in the group. They listened to one another—a little. They argued with one another, but did not usually concede acceptance of another student's point of view. They looked at one another's work, but might—or might not—be influenced by it. These students, it would have to be said, "think for themselves." Perhaps this should not surprise us; after all, what other form of actual thinking is there?

Group I—Dana, Stephanie, and Michael (May 30, 1990)

These children began by focusing on information that dealt with type of clothing and color to build up a representation of the problem situation.

> D: I'm just gonna draw a shirt . . . that's all we have to do . . . and then put like. [Dana drew three shirts.]
> S: I'm gonna make a shirt . . . and put white . . . wait . . . W for white. [She drew a shirt and placed the letter W inside the outline of the shirt.]
> M: Yeah, white shirt, white pants. [He drew a white shirt and white pants.]

They further refined their ideas by considering information about the number of items of clothing, drawing a picture that represented the data.

> S: Ok, blue and then a yellow shirt. [Stephanie drew blue and yellow shirts.] He has a pair of blue jeans and a pair of white jeans. [She drew two pairs of jeans.] How many different outfits can he make? Well . . . [Dana looked at Stephanie's paper and drew blue and white jeans.]
> M: He can make only two outfits.

The students decide that differences in the kinds of outfits are relevant. Michael's suggestion that there are two possible outfits stimulates Stephanie's curiosity. She cycles back and rereads the problem checking that the input data representation is consistent with her knowledge representation.

> S: Well, no [read] how many different outfits . . . he can make a lot of different outfits. Look, he can make white and white . . .
> D: He can make all three of these shirts with that outfit. [Dana pointed to her three shirts.]

From this language we might infer that Dana has the key idea for exhausting all possible combinations, but one should not be too quick to reach this conclusion. We cannot say, this early in our observations, whether Dana is building on her actual experience with "combinations of clothing items" being used to create "different outfits" or whether she is using a more abstract notion of what constitutes an "outfit." Figures 1a, 1b, and 1c show the final written work of Dana (Fig. 1a), Stephanie (Fig. 1b), and Michael (Fig. 1c). Notice that Stephanie uses her diagram to develop a coding strategy to form her combinations:

> S: I'm gonna make a shirt [Stephanie began to draw a shirt.] and put white . . . wait . . . W for white. [She then labeled the shirt by writing W inside its traced outline.]

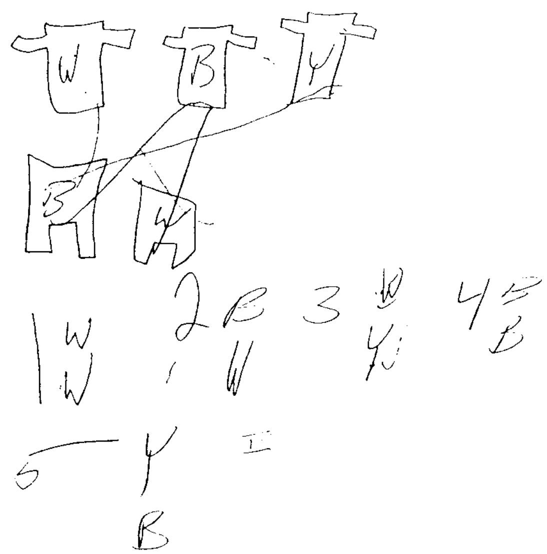

Fig. 1a. Dana's second grade solution.

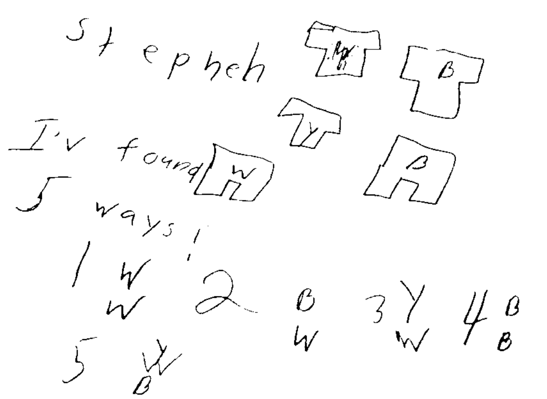

Fig. 1b. Stephanie's second grade solution.

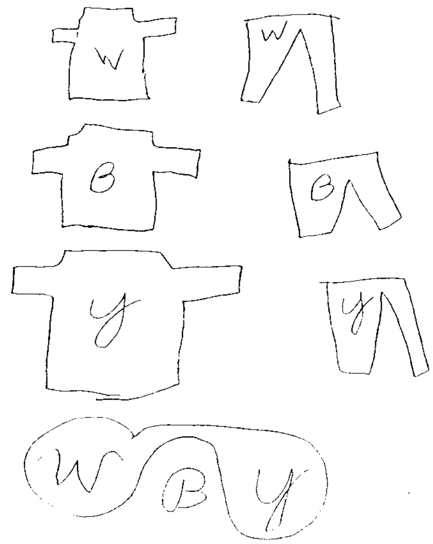

Fig. 1c. Michael's second grade solution.

Stephanie then illustrated each distinct outfit with a pair of letters, the first for the shirt and the second for the jeans. She recorded each outfit and kept track of them by numbering each combination. As she was recording the first outfit, she turned to Dana and Michael and said:

> S: You can make it different ways too. You can make white and white, that's one . . . W and W. [She drew a 1 and W over W, meanwhile Michael looked over at Dana's work.]
>
> M: That's what I'm doing. [He erases a piece of his white pants.]
>
> S: Two could be blue...blue jeans and a white shirt . . . blue, W. [She drew a 2 and B over W.]
>
> D: Yeah, well just put white with blue [Dana drew 5 connecting lines between the rows of shirts and pants.]

Dana found a notation that enabled her to make use of her original idea and spontaneously drew lines that connected each of her white and blue shirts to each of her blue and white jeans and her yellow shirt to her blue jeans. She concluded, as indicated in Fig. 1a, that there were a total of five different outfits. Stephanie continued to list pairs of letters for her outfits and concluded, also, that there were a total of five combinations.

S: Ssshhh . . . Ok, yellow shirt . . . number three can be a yellow shirt. [She drew 3 and Y over W.]

D: It can't . . . yellow can't go with the white.

Now, suddenly, we get a revelation! Dana is *not* thinking in terms of some abstract notion of what constitutes a "combination" or an "outfit" ! *She is building on her actual past concrete experience with selecting clothing items that can "go together" in a harmonious way.* (After all, when we say "I love your outfit," we are talking about the way things fit together, and not merely upon the fact that they are present.)

We get further data from Dana's drawing (Fig. 1a), which indicates that she didn't draw a line between the yellow shirt and white jeans. For Dana, an outfit is the kind of combination of clothing items that her experience has taught her to consider appropriate. She appears to ignore Stephanie's remark that the outfit doesn't have to match. Dana has not moved to the stage of thinking about abstract outfits which are to include every possible combination of one shirt with one pair of jeans, however unsightly the result.

During this time, Michael worked quietly alone, occasionally stopping to listen to his classmates, or to talk about his recording of white shirt with white pants and blue shirt with blue pants.

S: No . . . how many outfits can it make . . .?

M: I'm doing white shirt and white pants, blue pants, blue shirt.

S: It doesn't matter if it doesn't match as long as it can make outfits. It doesn't have to go with each other, Dana!

The videotape shows Dana tapping with her pencil and counting her five connecting lines.

D: Four outfits it can be . . . .

S: It can be more if you put them mixed up. Just watch. I'm on my third one right here . . . number four. It could be a blue shirt and a blue pants. [She drew 4 and B over B.] Number five can be . . . a white shirt . . . [She drew 5 and W] . . . Wait . . . [She erased the W.] OK . . . a blue shirt . . . Wait! . . . Did I do blue and white? [Dana looked over Stephanie's shoulder and Michael drew a blue shirt.]

Note that we are able to observe some of Stephanie's self-monitoring or "control" planning, as she checks over whether she "[did] blue and white."

Dana chose not to revise her picture to include the yellow shirt and white jean combination, even after Stephanie commented on the irrelevance of taste. In choosing not to follow Stephanie, Dana is exhibiting typical behavior for these students: They may listen to others, but they are not quick to change their minds. (Of course, if one construes outfit in the terms of everyday experience, as Dana was doing, then taste never is irrelevant.)

Nor had Stephanie herself found all possible combinations, despite the fact that she intended to do so. Her diagram did not include one combination, the white shirt and blue pants outfit. Notice in Fig. 1b that Stephanie's fifth entry shows a letter W between the letters Y and B as she recorded the yellow shirt and blue pants outfit. Observation of the videotape episode indicated that Stephanie first wrote the W, erased it, and then wrote a Y over the B. Dana then recorded Stephanie's solution on her paper beneath her line diagram. At this time, Michael continued to draw without speaking. Continuing:

D. What's two? [Dana was inquiring about Stephanie's second combination in an attempt to compare results.]

S: [Stephanie was not acknowledging Dana's request, and continued with her fifth combination.] It can be a yellow . . .

D: What's two? [She looked over Stephanie's shoulder and began to record her coded combinations. Michael looked up and continued to draw.]

Recall that early in the problem session, Michael reported that he had found two outfits, white shirt with white pants and blue shirt with blue pants. Stephanie tried to convince him that there were more combinations. However, Michael was engaged with drawing his own picture (Fig. 1c), and although he seemed to be aware of Stephanie's coding strategy, he appeared to reject it. Now we see Michael acknowledge Stephanie's strategy, but decide to reject it:

S: Two is blue shirt and white pants . . . a blue shirt and yel . . . wait . . . a yellow shirt . . . did I do yellow and white? A yellow shirt and blue pants. [Stephanie drew Y over B.]

D: A yellow shirt and blue pants.

M: I don't want to do it that way . . . I want to do it this way. [He referred to his own picture, and explicitly rejected Stephanie's system of coding.]

D: Well, do it the way you want.
S: Do you know what? There's five combinations . . . there's only five combinations. 'Cause look you can do a white shirt with white pants . . . .

Unsurprisingly, Michael's final solution does not resemble Stephanie's. In fact, what is particularly interesting about this classroom episode is that each child produced an independent solution, and each seemed to be satisfied with his or her own solution. Michael's solution is also idiosyncratic in a second way: He has shown three colors of jeans, instead of the two that the problem statement specified. This is a familiar kind of error; we conjecture that a partial representation—in this case, for the three colors of shirts—is inadvertently used in places where it should not have been (in this case, by applying it also to the jeans).

Group II—Jaime, Jeffrey, and Brian (May 30, 1990)

These children began by immediately deciding that they could make two outfits out of the five pieces of clothing.

J: Two 'cause he has a white shirt and a blue shirt and a yellow shirt and a blue jeans and a pair of white jeans. There's jeans and a shirt and jeans and a shirt.
Ja: Two.

Brian expressed concern about the shirts and pants matching.

B: A white shirt and a white pair of pants match, and a blue shirt and a blue pair of jeans match.

So! Brian, too, is building "metaphorically" upon actual previous experience. For Brian, as earlier for Dana, outfits need to match in order to be considered. This is what outfits are in the real world. One might have expected that the children would deal immediately with the abstract idea of "*any* combination of a shirt with a pair of jeans." But if one is aware of the importance of premathematical ideas one is not surprised to find that an outfit is what the child's experience has established it to be—it must match!

Jaime, too, is building upon the real world idea of an outfit:

Ja: Yeah, blue and yellow . . . blue and white, white and yellow go . . . white goes with everything!

Hence, she opened up the possibility of other combinations to which Brian agrees.

B: [To Jeff] Yeah, white goes with everything.

Although the question of what constitutes an outfit may not be settled—the children do not all agree—group II now begins to consider the critical question of how to be sure that you are finding every combination—however inclusive you wish to be:

Ja: Oh, I know how we can do it. [She was trying to find a systematic way of organizing the work.] White shirt with a blue pants . . .

Jaime (Fig. 2b) began to write, and Jeff and Brian stopped to observe what she was doing. Jeffrey (Fig. 2a), then counted the combinations as Jaime listed them verbally.

J: One . . .
Ja: Then a blue shirt with white pants . . .

To form the second outfit, Jaime reversed the two colors.

J: Two . . . that's two . . .
Ja: White shirt with white pants . . .
J: Three . . .

Jaime's strategy of reversing the pattern began to get into trouble when she reached the white and white combination and she could no longer switch top and bottom colors; a worse problem was coming next:

Ja: Yellow with the blue . . .
J: Four . . .
Ja: Blue with the yellow . . .

When Jaime noticed there were no yellow pants, she terminated her reversal pattern. Nonetheless the children continue their search:

J: Five, six, seven! Seven different pairs! First there's white and blue that's one.
B: How we gonna write this down?

Here we see explicit discussion of the question of inventing an appropriate written representation; of course, the matter of inventing notations has been an important part of this entire episode. Characteristically, each child has invented his own notation, with a modest amount of borrowing ideas from the others.

Notice that the children deal simultaneously with many different matters. This is an almost universal property of classroom discussions of this type. One consequence is that the children's remarks jump from one aspect of the problem to another, then frequently jump back to an earlier matter. Indeed, at this point Brian returns the discussion to the question of whether an outfit has to match. This time, though, Jaime and Jeffrey try to convince Brian that the outfits need not match:

B: A yellow shirt doesn't match.
Ja: Even though it's weird . . .
B: Yellow white . . . yellow blue . . .
J: You have to make it every single . . . anything . . . any combination . . . you have to make all the different combinations with everything.

So we see that Jeffrey is using the abstract idea of every possible combination, whereas Brian still wants the real-world meaning of outfit. Jaime seems to be somewhere in the middle—"even though it's weird," surely a real-world criterion still being employed.

Communication and Sharing

Almost immediately after this discussion, Brian and Jaime are observed watching and listening to the conversation of the Stephanie, Dana, and Michael group. Jaime and then Brian respond.

Ja: They drew it! [She looked at Dana, Mike, and Stephanie.]

Jaime remarked that the other group drew a diagram.

B: They say five combinations!

The children continued to work on their solution:

J: White shirts and white pants . . .
Ja: No, I have a white shirt match a white pair of pants, a blue shirt match a blue pair of pants, a yellow shirt match a white pair of pants . . . .
J: White pants and white, yellow pants and a [unclear] shirt.

Jaime then turned to Stephanie's group and said:

Ja: You still have to write about it!

Stephanie responded with the following interpretation of their task:

S: You don't always have to write about it . . . you can write a picture or a graph.

As the children continued to list their combinations, Jaime and Brian appeared to lose interest in recording all possibilities. An examination of the written work of each child (see Figs. 2a, 2b, and 2c) suggests no evident pattern for generating the combinations, although earlier Jaime had referred to an "opposites" pattern in her discussion with Brian and Jeff. Their strategy appeared to be one of writing whatever came into your head, which in some cases resulted in repeated outfits and the absence of checking for all possibilities. At this point in the exploration, Jeffrey decided to share his combinations with Jaime and Brian.

J: 1, 2, 3, 4, 5, 6. I got 6. I got white and white, blue and blue, yellow and blue, white and yellow, yellow and blue, and white and blue.

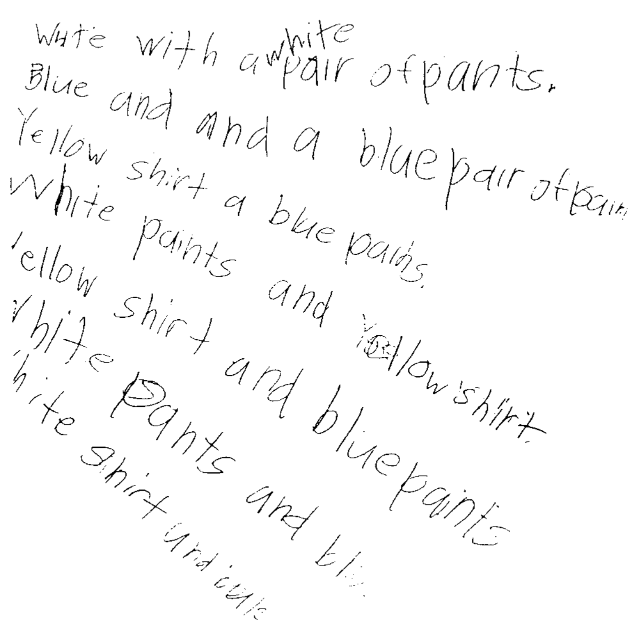

Fig. 2a. Jeffrey's second grade solution.

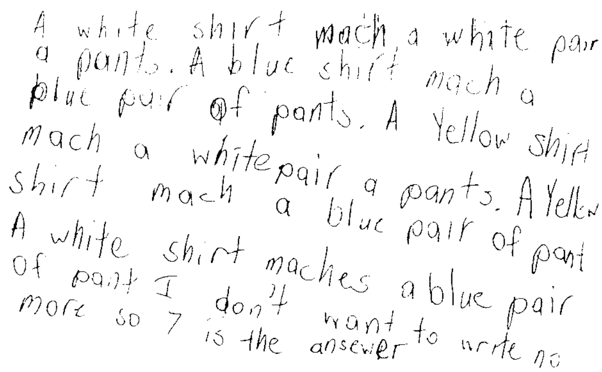

Fig. 2b. Jaime's second grade solution.

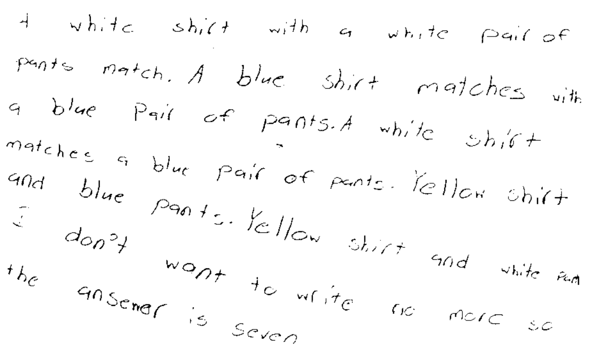

Fig. 2c. Brian's second grade solution.

Notice that Jeff's six outfits were not all different; the yellow shirt and blue pants combination (Fig. 2a) was repeated. Although Jeff read his solution aloud, he nonetheless failed to detect the repetition of the "yellow and blue" combination.

Ja: I did half of yours already. I put a white shirt matches a white pair of pants, a blue shirt matches a blue pair of pants, a yellow shirt matches a white pair of pants, a yellow shirt matches a blue pair of pants . . .

As Jaime (Fig. 2b) read her solution aloud, we observed that she, like Jeffrey, made use of the doubles pattern of blue and blue and white and white. However, she proceeded systematically to match the yellow shirt with all possible pairs of pants. At this point, the instructor (I) approached the children.

J: How 'bout white and blue?
Ja: A white shirt matches a white pair of pants.
I: How many did you get?
Ja: I got five!
B: I got six.
I: Ok, you gotta find them all.
Ja: I give up. [She did not write any more combinations.]

Jeffrey showed persistence in completing his list with the replication of the yellow shirt and blue pants combination. His seventh combination was the white shirt and blue pants. Jeffrey's two partners tired of the listing process, and waited for him to complete the list. Brian expressed his loss of interest with this process:

B: Let's write, I don't want to write no more so the answer's seven.

Effect of Sharing and Communicating within a Small Group

The most evident aspect of the influence of one child's work on another child is that this influence—at least on the surface—is surprisingly small. Michael looks at Stephanie's work and says: "I don't want to do it that way." Figure 2b displays the written work of Jaime, who, like Brian, found five combinations and wrote "I don't want to write no more so 7 is the answer." Note that seven was the answer of another child, so Jaime is in effect saying "OK, have it your own way; I don't really care." This is not agreement, and certainly not an intent to change her own thinking. It is withdrawing from the contest, renouncing the goal of thinking about the matter and trying hard to understand it. Perhaps Jaime, having found five combinations, became uncomfortable when Jeff indicated that there were seven. Jeff's strategy of reversing the order in which pants and shirts appear in his phrases may also have disarmed Jaime and discouraged her from pursuing her own strategy. Another possibility is that having found five combinations, Jaime had an understanding of how to obtain all combinations and simply ceased to record, letting Jeffrey finish the problem.

Whole-Class Sharing of Solutions in Grade Two

Following small group work (from which the preceding transcripts were taken), usual class procedure called for a total class meeting for students to share their solutions. Here, too, individualism prevailed. No singular strategy was adopted for use by the children and no agreement evolved as to the correct number of outfits. Some of the answers reported by different groups were three, five, six, and seven outfits. Generally, the children were pleased to share their representation of the problem solution and talk about how it was derived. Although numerous paths of solution were shared, the apparent result was that no one solution was widely accepted by the students, nor any one "correct" answer.[4]

GRADE THREE

The next opportunity to return to this same problem came five months later, in October, when the children were in grade three. Because both teachers were involved in the study, we know that there had been no class consideration of the problem in the interim.[5] This is of great importance, because one matter this study sought to explore was what happens to ideas in a child's head in the absence of explicit instruction.

Notice what has happened. The children have been given something to think about—first of all, of course, there is the problem itself. Second, there is the influence of the ideas of all of the other children. Even though most of the suggestions from others seemed to be rejected at the moment ("I don't want to do it that way!"), the suggestions themselves may have lingered in the mind, and over the ensuing months they may have taken a firmer shape. If we return to this identical problem now, five months later, will the students show evidence of having somehow incorporated any other children's ideas into their own thinking?

The grade three activity will be reported for the two pairs of students: group III consisting of Dana (D) and Stephanie (S) and group IV consisting of Jaime (Ja) and Michael (M).

Group III—Dana and Stephanie (October 11, 1990)

The following dialogue showed Dana and Stephanie beginning to build a representation for the input data. Dana read the problem aloud. Stephanie responded by suggesting that they draw a picture.

D: Stephen has a pair of blue jeans and a pair of white jeans. How many different outfits can he make?

S: We . . . why don't we draw a picture?

As Dana and Stephanie drew their pictures (Figs. 3a and 3b), we saw them focusing on the pieces of data that dealt with numbers of shirts and pants and their colors. In so doing, they searched for a way to map their knowledge representation to the data representation of the problem:

D: OK . . . He had a white shirt. [The girls drew pictures of shirts.]

S: So, I'll make a white shirt. [Notice that Stephanie was checking her representation as she drew her picture to match the problem data.]

D: A blue shirt . . .

S: I think I'll have to use the big marker for this one . . . you know, color it in blue. [Stephanie made explicit reference to shading the shirt blue to match the problem data.]

D: And a yellow shirt. [The girls drew another shirt.]

At this point, Stephanie (Fig. 3b) suggested that the data be coded by assigning the first letter of the color rather coloring the piece of clothing. Notice that she did not write what she said.

S: Why don't we just draw a Y, a B and a Y [sic] instead of coloring it in?

D: That's what I'm doing.

S: W, B, Y. [Stephanie placed a letter in each diagram of the shirt to represent its color.] OK, he has . . .

D: A blue.

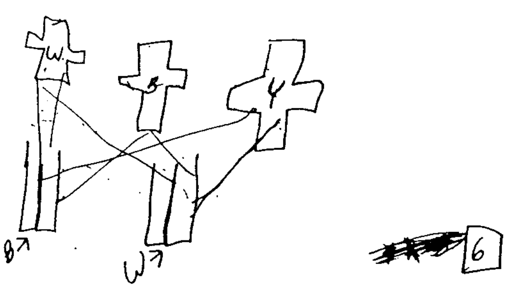

Fig. 3a. Dana's third grade solution.

[4]The lack of closure in reaching a single agreed-upon solution was, in a sense, deliberate. In our classroom work in schools we often find that students may not be ready to agree on a solution to a particular problem task. Rather than reach closure at the time and push for (or even present) a solution, our usual choice is to revisit the same problem at a later time.

[5]Nor had any related material been discussed.

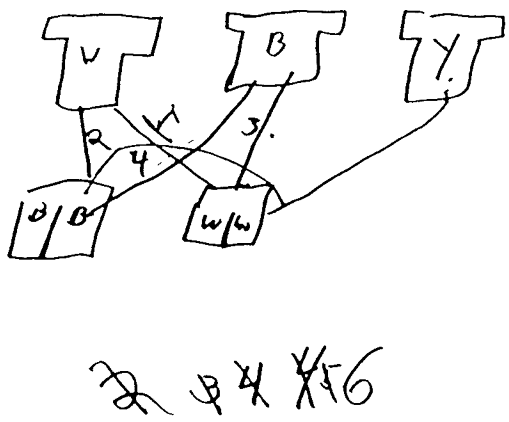

Fig. 3b. Stephanie's third grade solution.

Stephanie reread the problem as she worked to build a representation of the problem.

> S: Let me read this . . . . He has blue jeans and a pair of white jeans. OK . . . So let's make blue, blue, blue . . . [She was considering all possible outfits she could make with the blue jeans.]

Dana (Fig. 3a) monitored Stephanie's work, and completed the construction.

> D: And a pair of white jeans.

Stephanie indicated that she had begun to construct a representation of the relevant knowledge and proceeded to attack the problem.

> S: All right. Let's find out how many different outfits you can make. Well, you can make white and white so that would be one. I'm just going to draw a line . . . . [Stephanie rather than Dana was the person to initiate the connecting line strategy and drew a line from each shirt to the blue pair of jeans then from each shirt to the white jeans.]

Later, Dana and Stephanie were asked by the instructor why they used connecting lines. Stephanie replied:

> S: So we could make sure that we were . . . so that we didn't do that again [She referred to repeating a combination.] and say that was 7, 8, 9, 10 [She referred, again, to repeated combinations.] we drew lines . . . so then we could count our lines and say, "Oh, we can't do that again!" or so we could know if we already matched that . . . so we don't go . . . "Oh, OK, that's two [She drew the lines with her finger on the desk top to indicate the two combinations with the white shirt.], that's 4 [She indicated two combinations with the blue shirt.]...and we'd get more than we were supposed to.

Stephanie, in her justification of the use of the line strategy, indicated a shift from working with the representation of the problem data to working instead with a representation of the process by which she solved the problem. Her reflection on that process showed a development to another level of awareness, a shift to a meta level. Whereas at first her pictures represented direct translations of problem data (pictures and colors of shirts and pants), she now invented notation to monitor her own behavior. Notice that her diagram indicated a number label attached to her connecting lines which enabled her to keep track of each combination (Fig. 3b). In this session, neither girl used a coded listing strategy; they simply drew the three shirts and two pairs of pants, connected shirts to pants with lines, and counted their lines. Stephanie explained her preference for the use of lines because they indicated the number of combinations.

We would argue that the meta-concerns that the videotape records are not psychologically different from the direct attempts to draw shirts and pants and to deal with the original problem. The same processes of trying to identify, clarify, and analyze the problem are still in play—it is just that the problem itself has changed. At first, the problem was the one posed by the teacher, about making up outfits of shirts and jeans. Later on new problems have come to occupy the children's attention: how to organize their own work, how to know that they have not omitted any combinations, nor counted the same outfit more than once. The specific tasks are different, but the processes of dealing with them are basically the same.

Group IV—Jaime and Michael (October 11, 1990)

Figures 4a and 4b show the methods of solution used by third-graders Jaime (Fig. 4a), who had previously worked with Jeffrey and Brian, and Michael (Fig. 4b), who previously worked with Dana and Stephanie. The videotape indicated that Jaime

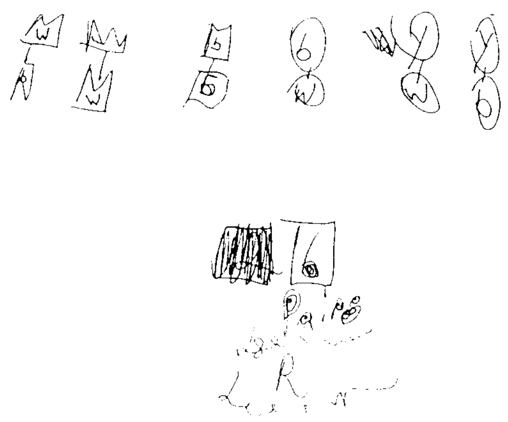

Fig. 4a. Jaime's third grade solution.

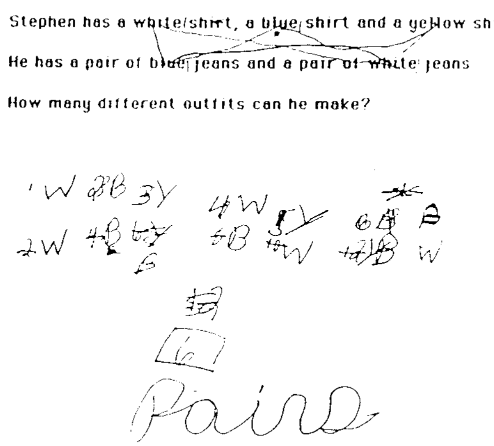

Fig. 4b. Michael's third grade solution.

and Michael worked individually, once again appearing not to listen to each other, each pursuing their own solution. Recall that in second grade: (1) Jaime had recorded her outfits as descriptive phrases, and lost interest after recording five combinations; and (2) Jaime had verbally noted Stephanie's method of drawing a picture to solve this problem.

Now, in grade three, the performances of the children are different. Each child seems indeed to have learned from the others (even though at the time they had seemed not to). Now Jaime draws a diagram (Fig. 4a) that represents each outfit with a two-letter code for color, she shows six shirts and six pants and draws a line to define each outfit as separate from the others. She uses a pattern to generate her diagram which matches each color shirt to both colors of pants before moving on to the next color shirt.

In grade two, Michael's written solution (Fig. 1c) displayed a diagram with three shirts (B,W,Y) and three pants (B,W,Y), with no connecting lines and no numerical answer. In grade three, Michael (Fig. 4b) employs a modified version of the connecting line strategy originally introduced by Dana in grade two (Fig. 1a), and applies it directly to the words in the stated problem. He does this by tracing his finger along the words in the problem to obtain his combinations. However, he records his answers with one-letter color codes, much like those used by Stephanie (Fig. 1b) in grade two. Although it was not obvious that Michael was attending to the representations of his classmates in second grade, his solution strategy in grade three provides strong evidence that he had been somehow aware of their work. Using his new-found strategy, he now achieves all six combinations. Similar results (leading to correct solutions) are shown for all of the children, as a detailed analysis indicates (Maher, Martino, and Davis, 1992a).

WHAT DO THE VIDEOTAPES SHOW?

In general, taped recordings of what students actually do reveal a very great complexity within what was once thought of as the "simple" world of "doing mathematics." Although one can find a seemingly endless collection of themes in these tapes, we will call attention to only four:

Thought in the Absence of Teaching

On May 30, 1990, the children were asked how many different combinations could be made from three shirts and two pairs of jeans. No consensus was reached about the correct answer. The teacher did not insist upon a class consensus, she did not criticize nor evaluate, and she showed neither the correct answer nor any method for solution. Nor did she deal with other similar or related problems. She left the students free to think about the matter if they chose to do so.

On October 11, 1990, the same children, now organized into different small groups, were given the same question. There had been no teaching of this topic in the meantime. Yet now every student was able to solve the problem, and each felt confident of their answer.

We do not suggest that one should never teach, nor do we suggest that the teachers did nothing. On the contrary, the teachers gave the students "tools to think with." First of all, there was the problem itself. It was intelligible and could be thought about. Second, there was the experience of working on the problem, which brought the students face to face with the need to keep track of their work and to organize it in some systematic fashion in order to make sure nothing was left out and nothing counted twice. There was the input from the other students, which seemed to be rejected, but in fact was not. Especially important, there were the "premathematical ideas" with which the children were already very familiar: shirts, jeans, combinations, counting, different vs. same, and so on. Every child was well able to think in terms of these ideas.

Building Representations in Your Mind

One of the main activities in mathematics is the building up of representations in our minds. What do we do when we listen? We try to build up in our mind a replica (as nearly as possible) of some representation that is in the speaker's mind. This is a very difficult process, as nearly all of our videotapes show only too clearly. In May, when the children seemed not to listen to one another, it would be more accurate to say that they had little success in building up in their own minds replicas of the representations being developed by the other students. They did input something, but it was only five months later that they demonstrated success in building up major parts of these representations. Note that most common theories of teaching pay little heed to the time required to build representations and often operate in ways that make representation building difficult or even impossible (see, for example, Davis, 1987, pp. 110–111).

There is the further question of the premathematical building blocks from which representations are constructed, and the distinction between metaphors based on experience (which probably have an essential role to play, and probably must not be bypassed) vs. subsequent abstract ideas that are constructed after one has used metaphoric assimilation paradigms for an adequate length of time. For some of these children, a true outfit had to represent a harmonious match of colors; only later did they come to the idea of putting things together in every possible way, no matter how unsightly the result. This, after all, is a far more abstract idea (and one that is hardly used in the everyday world).

Questions Left Open and Revisited

During the small-group discussions, we saw time and again that a question would arise, receive some attention, be left unresolved, and be taken up later on (and, once again, be left unresolved). Students seem to have a very large capacity for setting aside unresolved questions, to return to them later, perhaps many times. (Thus, different notational conventions came up and received some thought, with no consensus being reached. The matter would arise again, but—once again—no consensus would be reached. The question of "Must it match?" received this same treatment.) This process of leaving open is probably one more consequence of the difficulty of fitting input data into your own mental representations, which sometimes cannot be done instantaneously.

Capitulation without Agreement

One episode on the tape is perhaps especially interesting. Jaime and Brian, after recording five of the outfits, decided that they did not want to continue. While on the surface it may appear that they were accepting Jeffrey's answer of seven ["I don't want to write no more so the answer is seven."] there are other possible interpretations. Did the students believe the answer was seven? We hardly think so. From their tone of voice and demeanor, it appeared that they were unwilling to pursue what they had come to consider a tedious task. They had had enough! Their written work and conversation indicated that they had begun to understand the problem and were developing a heuristic for carrying it to completion. How is a teacher to deal with this situation? Teachers who believe that learners must build up ideas in their own heads may conclude that insisting upon closure may, in the long run, be ineffective. Others may feel obligated to encourage students to reach closure.

DEPTH OF VIDEOTAPED CONTENT

Videotaping, followed by careful analysis, allows us to see subtleties in student thought processes and to track the development of student thinking over extended periods of time. The depth of content that can be studied in this way is quite remarkable. For this level of detail a price must be paid in terms of time and effort required for analysis. A particularly impressive instance is given in Schoenfeld *et al.* (in press), where a team of researchers spent one and a half years analyzing a three-hour videotape of a single student.

CONCLUSIONS

Mathematics teaching used to be thought of in terms of a teacher standing in front of a class, giving lectures, directions, or explanations to students—or possibly supervising seat work—and methods for studying this process are fairly well developed. In recent years scholars have come to think of mathematics learning quite differently. In the first place, it is now seen as a matter of individual students building up ideas in their own minds (Davis *et al.*, 1990), often constructing abstract ideas by assembling previously learned experiential components (Davis, 1984; Papert, 1980). Social influences are seen as playing an important role (Vygotsky, 1978; Johnson and Johnson, 1991). Classrooms are often organized into small groups of students, working together for cooperative learning (NCTM, 1989). This calls for new emphases on what needs to be studied—what really happens with all of these students working together in small groups? How are ideas built up in the mind of an individual student, when this takes place at least partly in a classroom where other people are involved?

This also calls for a new technology. We argue that videotaping can play a major role in making possible the kinds of observations that are needed in this radically new situation, and in this report we have given several examples.

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