Reform in Primary Mathematics Education: A Constructivist View

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Sometimes in danger of getting lost amidst the many theories of instruction in mathematics, children must be encouraged to do their own thinking rather than being taught shortcuts.

Mathematics education in the United States is well known to be in need of reform.[1] This need for improvement is clear from the beginning of the elementary grades, as we can see in a report of the National Assessment of Educational Progress:

> One would expect a majority of nine-year-olds . . . to have mastered basic mathematical operations . . ., as these skills are usually taught in elementary school. The fact that only twenty-one percent of the nine-year-olds attained this level in the 1986 assessment . . . suggests that reform in the mathematics curriculum may be warranted from the earliest grades.[2]

The research and theory of Jean Piaget have demonstrated that children acquire number concepts by *constructing them from the inside*, in interaction with the environment, rather than by *internalizing* them from the environment as traditional mathematics educators assume.[3] On the basis of this theory, called constructivism, we have been developing a way of "teaching" arithmetic in the primary grades.[4] One of the conclusions we have reached is that the teaching of rules, or algorithms, to get correct answers is harmful to children's learning of arithmetic. Our reasons for saying this are that (a) these rules go counter to children's natural ways of thinking, (b) algorithms "unteach" the little understanding children have of place value, thereby depriving them of opportunities to develop number sense, and (c) the history of computational procedures suggests that children would understand algorithms better if they were allowed to go through a constructive process similar to this evolution.

This article elaborates on these points and explains why we believe that children must be encouraged to do their own thinking rather than be taught shortcuts. Curriculum reform requires that we study children's process of learning and facilitate their process of construction instead of continuing to teach in ways that seem efficient for adults.

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Algorithms and Children's Natural Ways of Thinking

Two of us work at Hall-Kent School, a public school near Birmingham, Alabama, where a primary mathematics program is being developed on the basis of Piaget's theory. In this program teachers do not introduce any algorithms but, instead, encourage children to invent their own procedures for the basic operations. The following examples are typical of the procedures children invent and illustrate how their natural ways of thinking are in direct opposition to that of conventional algorithms (from the higher-order units to the ones in addition, subtraction, and multiplication, and from the lower-order to the higher-order units in division). These procedures are all invented in mental arithmetic, and children use writing only when a problem has so many steps that it is impossible to remember the results of previous steps (e.g., the division problem below).

$$\begin{array}{r} 16 \\ \underline{+17} \end{array}$$

$10 + 10 = 20$
$6 + 7 = 13$
$20 + 10 = 30$
$30 + 3 = 33$

$10 + 10 = 20$
$7 + 3 =$ another ten
$20 + 10 = 30$
$30 + 3 = 33$

\+ + + + + + + + + +

$$\begin{array}{r} 43 \\ \underline{-15} \end{array}$$

$40 - 10 = 30$
$3 - 5 = 2$ below 0
$30 - 2 = 28$

$40 - 10 = 30$
$30 - 5 = 25$
$25 + 3 = 28$

\- - - - - - - - - -

$$\begin{array}{r} 13 \\ \underline{\times 4} \end{array}$$

$4 \times 10 = 40$
$4 \times 3 = 12$
$40 + 12 = 52$

× × × × × × × × × ×

$22 \overline{)275}$

$22 + 22 + 22 + 22 \ldots$ until the total comes close to 275, or

$10 \times 22 = 220$, and then proceeding by addition until the total comes close to 275

The order of steps in conventional algorithms is from right to left in addition, subtraction, and multiplication, and from left to right in division. Algorithms thus go counter to children's ways of thinking in each one of these operations and prevent them from using their own natural ability to think. Algorithms further hinder children's thinking in another way: By requiring them to write, algorithms focus children's attention on writing rather than on reasoning.

A Piagetian Task

Our emphasis on children's natural ways of thinking is based on Piaget's theory about the nature of logico-mathematical knowledge. The best way to explain this theory is with examples of children's reactions to a task.

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In this task, which was devised by Bärbel Inhelder and Jean Piaget, two identical glasses and about fifty wooden beads (or chips, beans, etc.) are used.[5] The child is given one of the glasses, and the interviewer takes the other glass. The interviewer then asks the child to drop a bead into his or her glass each time she drops one into her glass. After about five beads have thus been dropped into each glass with one-to-one correspondence, the adult says, "Let's stop now, and you watch what I am going to do." The interviewer then drops one bead into her glass and says to the child, "Let's get going again." The adult and the child drop about five more beads into each glass with one-to-one correspondence, until the adult says, "Let's stop." The following is what has happened so far:

Adult: 1+1+1+1+1+1+1+1+1+1+1
Child: 1+1+1+1+1 +1+1+1+1+1

The adult now asks, "Do we have the same amount, or do *you* have more, or do *I* have more?"

Four-year-olds usually reply that the two glasses have the same amount. When we ask, "How do you know that we have the same amount?" the children explain, "Because I can see that we have the same (amount)." (Some four-year-olds, however, reply that *they* have more, and when we ask them how they know that they have more, their explanation consists of only one word: "Because.")

The adult goes on to ask, "Do you remember how we dropped the beads?" and four-year-olds usually give all the empirical facts correctly, including the fact that only the adult put one bead into the glass at one point. In other words, four-year-olds remember all the empirical facts correctly and base their judgment of equality on the empirical appearance of the two quantities.

By age five or six (in kindergarten), however, most children deduce logically that the teacher has one more. When we ask these children how they know that the adult has one more, they invoke exactly the same empirical facts as the four-year-olds.

No one teaches five- and six-year-olds to give correct answers to these questions. Yet children all over the world become able to give correct answers by constructing numerical relationships through their own natural ability to think.

If a child says that the adult's glass has one more bead, the interviewer goes on to pose the next question: "If we continued to drop beads all day (or all night) in the same way (with one-to-one correspondence), do you think we will have the same number at the end, or will *you* have more, or will *I* have more?" Five- and six-year-olds divide themselves into two groups at this point. The more advanced group answers in the way that adults would; that is, there will *always* be one more in the interviewer's glass. The other group makes empirical statements such as "I don't know because we haven't done it yet" or "We don't have enough beads to keep going all day."

Children's responses in this task illustrate their construction of number concepts from within. No one teaches five- and six-year-olds to give correct answers to these questions. Yet children all over the world become able to give correct answers by constructing numerical relationships through their own natural ability to think. This construction from within can best be explained by reviewing the distinction Piaget made among three kinds of knowledge according to their ultimate sources and modes of structuring—physical knowledge, logico-mathematical knowledge, and social (conventional) knowledge.

Physical, Logico-Mathematical, and Social Knowledge

Physical knowledge is knowledge of objects in external reality. The color and weight of a bead are examples of physical properties that are *in* objects in external reality and that can be known empirically by observation.

Logico-mathematical knowledge, by contrast, consists of *relationships* created by each individual. For instance, when we are presented with a red bead and a blue one and think that they are *different*, this difference is an example of logico-mathematical knowledge. The beads are observable, but the *difference* between them is not. The difference exists neither *in* the red bead nor *in* the blue one, and

When children have not yet constructed the logico-mathematical relationship of numbers in their heads, all they can get from the experience is physical, empirical knowledge. This is why four-year-olds can remember the empirical facts of dropping all the beads except one with one-to-one correspondence.

if a person did not put the objects into this relationship, the difference would not exist for him or her. Other examples of relationships the individual can create between the same beads are "similar" and "two." Logico-mathematical knowledge is thus not empirical knowledge, since its source is in each individual's head. (However, we could not make relationships if there were no objects in the external world to put into relationship.)

The ultimate sources of social knowledge are conventions worked out by people. Examples of social knowledge are the fact that Christmas comes on December 25 and that a tree is called "tree."

The distinction among the three kinds of knowledge makes it possible to understand why most four-year-olds in the task described earlier say that the two glasses have the same amount. When children have not yet constructed the logico-mathematical relationship of numbers in their heads, all they can get from the experience is physical, empirical knowledge. This is why four-year-olds can remember the empirical facts of dropping all the beads except one with one-to-one correspondence. This one-to-one correspondence, however, is only empirical, and four-year-olds judge the quantity of beads also empirically. This is why they say that the two glasses have the same amount and explain, "I can *see* they have the same amount."

By age five or six, however, most children have constructed the logico-mathematical knowledge of numbers and can deduce from the same empirical facts that the teacher has one more bead. However, number concepts take many years to construct, and the child who has created them up to ten or fifteen does not necessarily have concepts of fifty, a hundred, or more.

Piaget's theory about the nature of logico-mathematical knowledge explains why we say that children have to construct this knowledge from within, through their own natural ability to think. Traditional mathematics educators are not aware of the difference between logico-mathematical and social knowledge and advocate the teaching of algorithms as if arithmetic were social knowledge. As a result, they unwittingly impose procedures that go counter to children's natural ways of thinking.

Algorithms, Number Sense, and Understanding of Place Value

The conventional algorithms for addition, subtraction, and multiplication are efficient because they allow us to treat every column as ones. These procedures are efficient for adults who already understand place value. For young children who are not sure of place value, however, these algorithms serve to reinforce their tendency to think about every digit as ones. Children who are made to use algorithms are thus deprived of opportunities to develop number sense. Below are three examples from research demonstrating that when children are encouraged to do their own thinking, they understand "tens and ones" better, have better number sense, and are superior in mental arithmetic.

Children's Explanation of "Carrying"

Individual interviews were conducted with eighty-five second graders to evaluate their understanding of "carrying" and of "tens and ones."[6] Thirty-nine of these children were in a traditional program in which algorithms were taught, and the other forty-six were in a constructivist program that encouraged children to invent their own procedures.

In the interview, each child was shown a card on which the following problem was written:

$$\begin{array}{r} 16 \\ +17 \\ \hline \end{array}$$

The interviewer asked the child to add the numbers mentally, give the answer, and then explain how he or she got the answer. Almost all the children in both groups gave the answer of 33. All the traditionally instructed children explained that they added the 6 and the 7 first. Almost all then said, in essence, "I put my 3 down here (pointing) and 1 up there, and 1 and 1 and 1 is 3; so I put 3 down here, and the answer is 33." By contrast, almost all the children in the constructivist group added the tens first and then the ones as described earlier.

The child was then given sixteen and seventeen chips for the two numbers and asked to explain, using the chips, how he or she arrived at the answer. The percentage of children who correctly explained "carrying" ten was twenty-three for the traditionally instructed group and eighty-three for the constructivist group ($p < .001$). The great majority of the traditionally instructed children "carried" one chip instead of ten, showing that they understood neither the algorithm nor "tens and ones" in this context.

Children's Handling of Misaligned Digits

In the final part of the interview, the child was given a sheet of paper on which the problem below was written.[7]

```
 4
 35
+24
```

The interviewer asked the child to "read these numbers" and then to write the answer. When the child finished, the interviewer asked him or her to read the answer aloud and then inquired, "Does that sound right?"

The percentage of children who wrote "99" by mechanically following the rule of adding the columns was seventy-nine for the traditionally instructed group and eleven for the constructivist group ($p < .001$).

Children's Mental Arithmetic

Different groups of second graders were individually interviewed more recently and asked to solve the following problem (written horizontally) in their heads: 7 + 52 + 186. As can be seen in Table 1, the second graders who had not been taught any algorithms (class 3) did much better than those who had been taught these rules (class 1). Forty-five percent of class 3 gave the correct answer of 245, compared to twelve percent of class 1. Most children in class 1 used a right-to-left procedure. Those in class 3, on the other hand, all used a left-to-right procedure and said, for example, "180 + 50 = 230, 230 + 6 = 236, 236 + 2 = 238, 238 + 7 = 245." It is therefore not surprising that the errors found in class 3 were much more sensible than those in class 1. In class 3, most of the wrong answers were between 235 and 255. In class 1, by contrast, many children got small totals such as 29 and large ones ranging from 838 to 9308! (Those who got 29 or 30 did so by treating all the digits as ones, i.e., 7 + 5 + 2 + 1 + 8 + 6 = 29.) More than half of the children who had been taught algorithms thus demonstrated a lack of number sense.

The children in class 2 were in between, both in the instruction they had received and in the results. The teacher of class 2 did not teach the algorithm for addition but taught the one for subtraction and did not stop parents from teaching these rules.

When children are encouraged to do their own thinking, they develop number sense, and it becomes unnecessary to teach algorithms, number sense, mental arithmetic, and estimation separately. In an age of computers and calculators, number sense, mental arithmetic, and estimation are particularly important because children have to be able to sense an error when they hit a key unintentionally. The teaching of written algorithms was appropriate when all

Table 1

Second Graders' Answers to 7 + 52 + 186

	Class 1 *n* = 17	Class 2 *n* = 19	Class 3 *n* = 20
	9308		
	1000		
	989	989	
	986		
	938	938	
	906		
	838	810	
			617
		356	
	295		255
			246
245	(12%)	(26%)	(45%)
			243
		213	236
	200	213	235
	198	199	
		133	138
		125	
	30	114	
	29		
	29		
I can't do this.	(12%)	(21%)	(20%)

we had was paper and pencil. In an age of calculators, however, ability to think and to estimate answers becomes much more important than ability to follow rules of written procedures.

The teaching of written algorithms was appropriate when all we had was paper and pencil. In an age of calculators, however, ability to think and to estimate answers becomes much more important than ability to follow rules of written procedures.

The Constructive Process in the History of Computational Procedures

As an epistemologist, Piaget believed that the way to understand the nature of knowledge is to study the history of its construction by the human species as well as by individual children.[8] Just as he found many parallels between the evolution of physics, astronomy, etc., and the development of children's thinking in these areas, we found similarities between the history of computational techniques and the increasingly more efficient procedures invented by children at Hall-Kent School. As revealed in the following statement, today's algorithms were a very late achievement in human history: "It was not until 1600 that our modern Hindu-Arabic decimal system of numeration became generally accepted as the standard system of computations, replacing the use of Roman numerals."[9]

Until this surprisingly late date, most of our ancestors performed their computations with objects such as pebbles and counters, and with abacuses. Following is a sketch of how these objects were used before they were replaced by writing.

The Use of Objects

The Roman calculation board shown in Figure 1a consisted of a frame with parallel columns. The first column, on the right, was for ones, the second for tens, etc., and pebbles or counters were placed in each column to represent, for example, 4,365 as shown in this figure.

A variety of boards, tablets, and abacuses have been invented, but the basic principle of representing the base-ten system remained the same for centuries. Figure 1b shows 4,365 with a horizontal system that used the space between the lines to represent fives. "Five ones" is thus represented with one pebble above the line for ones. "Six tens" is shown with one pebble on the tens line and one above this line. This use of five as an intermediate higher-order unit

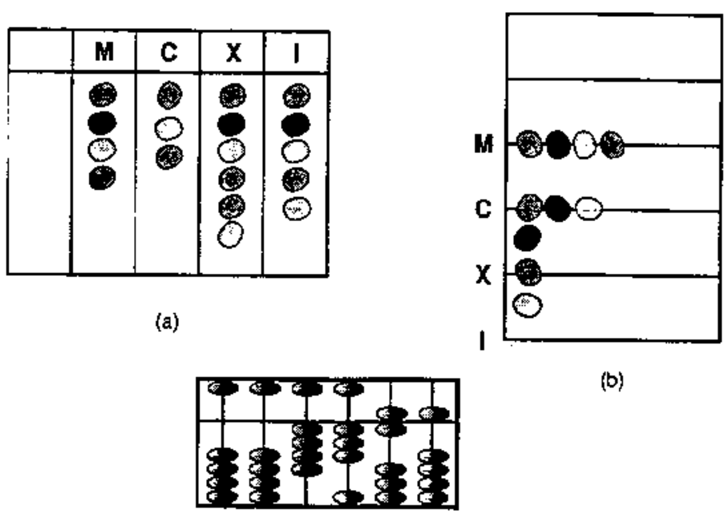

Fig. 1. Objects used as computational tools.

makes it easy to recognize six, seven, eight, and nine at a glance.

By putting ten beads that could be slid on a cord or a stick, our ancestors made the kind of abacuses that can still be seen today in many classrooms. The abacus disappeared from Europe by 1700 but is still being used in Asia. The most modern abacus being used in Japan today is shown in Figure 1c. The beads above the horizontal divider each stands for a five, and the ones below it each stands for a one. Figure 1c shows 4,365 represented by raising four ones in the thousands column, raising three ones in the hundreds column, lowering a five and raising a one in the tens column, and lowering a five in the ones column.

It is important for math educators to note that when our ancestors were using the abacus, they used writing to record only the *results* of the calculations carried out with the abacus. Groza states:

> Around 1100 the general public used Roman numerals and an abacus. Businessmen sat before a line abacus or counting table or "counter" (from which we obtain our present word *counter* as used in stores). Lines were ruled on the table [as shown in Figure 1b] to indicate the powers of 10 and loose counters were placed on these lines or between them and then moved as the calculations were performed.[10]

In Japan today, addition and subtraction are performed on the abacus by beginning with the highest-order unit and proceeding to the right, toward the ones. Instead of "carrying" and "borrowing," the abacus is used in the ways described below with respect to the following examples:

4,365	4,365
+ 987	- 987

To add 900, 100 is subtracted first and 1,000 is added. To add 80, 20 is subtracted first and 100 is added. To add 7, 3 is subtracted first and 10 is added. To subract 900, 1,000 is subtracted first and 100 is added. To subtract 80, 100 is subtracted first and 20 is added. To subtract 7, 10 is subtracted first and 3 is added. The reader must have noted striking similarities with the procedures invented by second graders at Hall-Kent School.

Compared to writing, physical actions on pebbles and beads are much more closely related to mental actions (thinking). In fact, these physical actions are direct extensions of mental actions. For example, pushing one bead up to add one is a direct extension of this mental action (whereas writing "+ 1" is not). The use of an abacus is also closely related to mental actions in another way: The person using an abacus has to know whether the place value is ones, tens, hundreds, etc. In a written algorithm, by contrast, once the columns have been aligned, every column can be treated as ones.

While the general public thus used counters and abacuses prior to 1600, a literate minority was inventing computational procedures that used writing. Much of this history has been lost, but a variety of procedures have nevertheless been preserved. We now turn to some examples, limiting ourselves to addition.

The Use of Writing

Five major procedures are described below: Bhaskara's method, the Hindu scratch method, and three others that include a method of "carrying."

1. Bhaskara's method (c. 1150).[11]

Groza describes the following way of adding 278 and 356:

Sum of the units	8 + 6 =	14	(14)
Sum of the tens	7 + 5 =	12•	(12)
Sum of the hundreds	2 + 3 =	5••	(5)
Sum of the sums		634	(634)

The column to the right in parentheses is the version given by Smith that does not use dots for zeros.[12]

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Pearson describes the following similar but "left-handed" method.[13]

```
 278
+356
----
 5
 12
  14
----
 634
```

2. The Hindu scratch method.[14]

This method presented by Groza proceeds from left to right. The sum is written at the top in steps as illustrated below for the 278 + 356:

```
            6          63
 5         52         524
278 ----> 278 ----> 278
356        356        356
```

When this computation was done on a 'dust' board, the digits were erased as they were used instead of being scratched out. The successive phases of the same procedure resulted in progressively less writing as can be seen below:

```
278 ----> 578 ----> 628 ----> 634
356        56         6
```

3. A left-to-right procedure described by Smith.[15]

```
 278
+356
----
 524
 63
```

4. A column-by-column procedure described by Pearson.[16]

```
 2      7      8
+3      5      6
--     --     --
 5     12     14
 6      3      4
```

5. Today's conventional algorithm of "carrying."

The following historical account of "carrying" by Smith (1925) helps us understand the conventional nature of this rule and the lateness of its invention:

> The expression "to carry" . . . probably dates from the time when a counter was actually carried on the line abacus to the space or line above, but it was not common in English works until the 17th century. Thus, we have Recorde (c. 1542) using "keepe in mynde," Baker (1568) saying "keepe the other in your minde," and Digges (1572) employing the same phraseology and also saying "keeping in memorie," and "keeping repose in memorie." The later popularity of the word "carry" in English is largely due to Hodder (3rd ed., 1664).[17]

While writing is removed from mental actions compared to physical actions on counters and abacuses, the old written procedures still allowed our ancestors to go through a careful process of reasoning step by step. In fact, similarities can be noted between our ancestors' written computational procedures and those invented by children at Hall-Kent School. Second graders at Hall-Kent School, for example, use writing to record mostly the *results* of their thinking. Our children also proceed from left to right as stated earlier.

We can see in light of the history sketched above that today's algorithms are the results of centuries of construction by the human species. The teaching of these algorithms is an attempt to transmit to children only the surface behaviors resulting from this historical evolution. Bypassing the constructive process may seem like an efficient way of teaching. However, children save time in the long run when they are allowed to invent their own shortcuts rather than being taught to mimick the final results of centuries of human inventions. When children are allowed to construct their own logico-mathematical knowledge, they invent increasingly more efficient procedures just as our ancestors did. By doing their own thinking and eventually accepting conventional algorithms of their own accord, children come to understand these rules and truly make them their own.

This article began by pointing out the need for reform in mathematics education. Attempts at reform must not aim at teaching adult algorithms "better." Instead, the time has come to take children's thinking seriously and to make fundamental changes. Instruction must enhance, rather than undermine, children's own construction of mathematics.

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