MATHematics illuminated

PARTICIPANT GUIDE

UNIT 13

UNIT 13

THE CONCEPTS OF CHAOS

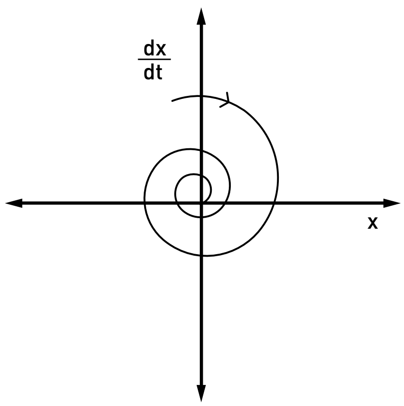
PARTICIPANT GUIDE

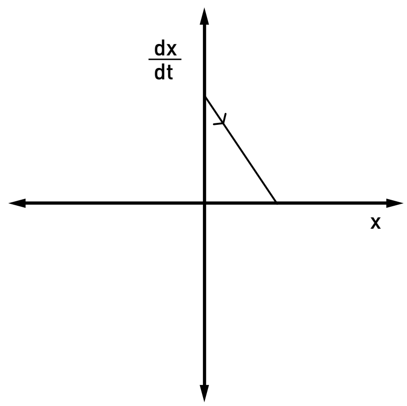
ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

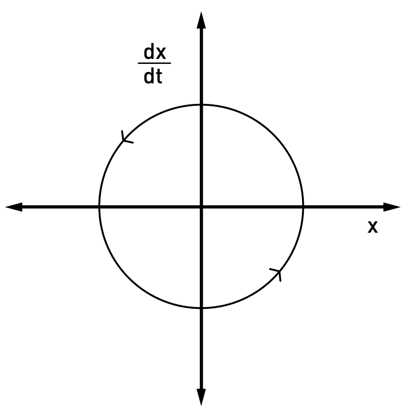
ACTIVITY 1

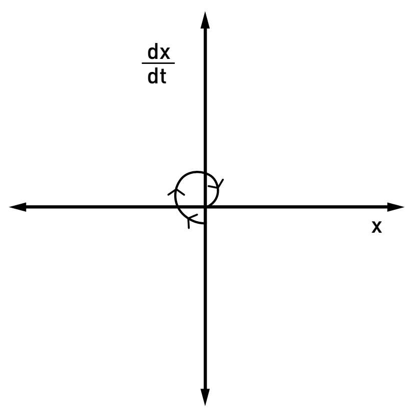
1. Phase space puzzle #1: One of the major tools used to analyze nonlinear systems is phase space. Recall from the text that a phase space is a graphical/pictorial representation of the behavior of a system. In your small groups, try to come up with a few situations that could be represented by the following phase spaces:





2. Phase space puzzle #2: In your small groups, try to come up with a few situations that could be represented by the following phase spaces:





ACTIVITY 1

3. In your groups, draw a possible phase space for each of the following scenarios, and be ready to explain your reasoning:

- A car driving at continuous speed
- A car accelerating from rest and then crashing
- A grape rolling off of an upside-down bowl
- A grape rolling in a rightside-up bowl.

ACTIVITY 2

THE FLUTTERBY EFFECT

MATERIALS

- 20 index cards or playing cards per group of three participants
- 9 feet of string per group
- A measuring tape or a meter stick for each group
- Masking tape
- Pennies

One of the most famous "examples" of chaos theory is the so-called "butterfly effect". This is the classic situation in which a butterfly in China, beating its wings, causes tiny changes in air pressure and currents that in turn cause larger and larger changes until, eventually, a tornado forms over Texas. This example, while vivid and memorable, is problematic in a number of ways, not the least of which is the fact that weather is a vastly complex system of many variables. It is perhaps not surprising that wildly unpredictable things can happen. True chaos can arise in very simple systems, imposing limits on predictability and exposing the extreme sensitivity of outcomes to tiny changes in initial conditions. What is remarkable is that although chaos may lead to unpredictable effects, these effects often are not "wildly" unpredictable—the range of possible outcomes is somewhat constrained.

In this activity you will see several hallmarks of chaos in the fluttering of falling index cards or playing cards.

With your group, find a circular space about 10 feet in diameter that has no obstacles. Make sure there is no wind or obvious sources of air disturbance. Tape one end of your string to the ground in approximately the center of the space (it doesn't have to be perfect). The string will be used as a way to calibrate your index-card-dropping location.

1. Drop a penny three times from a height of 2 meters over the point where you taped your string. Try your best to drop it from exactly the same position each time. After each drop, measure the distance between the target (where the string is taped) and the point where the penny landed. Describe your results.

2. Was there much variability in the results of the three drops?

ACTIVITY 2

3. Repeat the same procedure with one index card. After each drop, measure the distance from the target to the center of the index card. Compare and contrast your results with those obtained with the penny.

4. What are some possible explanations for the discrepancy between the results obtained with pennies and index cards?

Have the dropper drop all 20 index cards, one at a time, from a height of 0.5 meters. Try to drop each card from exactly the same position—the string can help you make sure you are directly over the target. After all 20 cards have been dropped, have the measurer measure the distance from the target to the center of each card while the recorder records the distances and finds the mean. Repeat this experiment twice more from the height of 0.5 meters.

RESULTS:

Now, repeat the same experiment from a height of 1.5 meters. Do three trials (dropping all 20 cards for each trial) and find the mean.

RESULTS:

ACTIVITY 2

Finally, try the experiment from a height of 2.5 meters. Again, do three trials and find the mean.

RESULTS:

ANALYSIS

1. What do you observe about the distribution of the cards around the target for each height?

2. How did statistical outliers affect the mean for each height?

3. For each height, how did the mean distance from the target compare to the results obtained by groups around you and the class as a whole?

4. Explain the role of chaos theory in what you observed. Talk about sensitive dependence on initial conditions and limits on predictability.

ACTIVITY 3

MATERIALS

- Poster paper
- Markers
- Tape

In the text you found a brief description of one particular way to think about chaos theory in a very simple system—the "multiply by ten and chop off the integer" model of kneading dough. Recall that the multiplying by ten is like stretching the dough and taking the non-integer remainder is akin to chopping off a bit of dough after each stretch.

In your group, design a poster that uses this metaphor to explain the essential features of chaos theory. Your poster should:

- Demonstrate sensitivity to initial conditions (show how this model can take two very similar starting values and end up with very different ending values).

- Show a limit to the variability of results (the end results can be far away from each other, but not arbitrarily far away from each other).

- Explain the role of iteration in chaos theory.

Feel free to use examples, explanations, and metaphors to make your poster come alive.

ACTIVITY 4

ORDER IN THE CHAOS

MATERIALS

- One six-sided die for each group of three
- Paper
- Ruler
- Transparency film
- Overhead markers
- Graph paper
- Colored pencils

One feature of many chaotic systems is their relationship to fractals. When graphed in phase space, the orbits that represent the evolution of chaotic systems tend to be fractal in nature—that is, "zooming in" brings more and more detail and structure into view, not less. In most real-world applications, the fractals generated by orbit diagrams of nonlinear systems are not immediately recognizable as fractals—as the Koch snowflake is. In this activity, we will use a very simple chaotic process that combines chance with some simple rules to plot the path of a particle.

a) On a sheet of transparency film draw a large triangle and label the vertices A, B, and C.

b) Mark a fourth point outside of the triangle; this is the initial position of your particle.

c) Roll the die. If it comes up a 1 or a 6, go half the distance to point A (from your initial position) and mark a new point. If it comes up a 2 or a 5, go half the distance to point B and mark a point. If it comes up a 3 or a 4, go half the distance to point C and mark a point.

d) Roll again, and starting from the newly marked point, follow the same rules to locate and mark another point.

e) Do this about 40 times, and sketch the shape of the path that emerges.

f) Repeat the process two more times, each time on a new transparency film and each time with a different starting position. Make sure, however, that the triangle is the same size and shape in each trial.

ACTIVITY 4

g) When you have finished three trials, overlay your transparencies and describe the result.

h) What happened when you changed your starting point?

This is a famous fractal known as a Sierpinski gasket. It has the quintessential fractal property of self-similarity at different scales. We saw this fractal in the activities for Unit 5.

i) Write an alternative, iterative replacement rule that would produce the same shape. For an example of such a rule, see the discussion on the construction of the Koch snowflake in Unit 5.

j) Use the definition of fractal dimension given in Unit 5 of the text to find the dimension of this object.

BONUS I:
Write the first 7 or 8 rows of Pascal's Triangle. Color the odd numbers one color and leave the even colors blank. What have you created?

BONUS II (fun fact):
Because of its exquisite detail at multiple scales, the Sierpinski gasket serves as a model for many high-frequency radio antenna designs, such as those used in GPS receivers. It is, of course, impossible to construct a real Sierpinski gasket, because there is a lower limit to the detail that can be created with atoms.

CONCLUSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

DISCUSSION

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 13, The Concepts of Chaos could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no “frontloading?”

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards?
Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

NOTES