MATHematics illuminated

FACILITATOR GUIDE

UNIT 9

UNIT 09

GAME THEORY

FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

ACTIVITY 1

(15-20 minutes)

THE PRISONER'S DILEMMA

In the text for Unit 9, you learned about the classic game theory scenario of the Prisoner's Dilemma. Prisoner's Dilemma is a non-zero-sum game in which rational players will each play a strategy that leads to a sub-optimal result. In your groups, brainstorm as many real-life situations as you can that can be modeled as "prisoner's dilemmas." For each situation, identify the relevant payoffs and strategies and, if possible, create a payoff matrix. Be ready to share your ideas with the larger group.
Answer: Answers will vary, but here's an example. In the classic game of "chicken," two drivers drive head-on toward each other attempting to make the other person swerve. The person who swerves loses his (or her) car to the other person. The strategies for each player are "swerve" or "don't swerve," and the payoffs are as follows (assume that both cars have airbags so that no one dies in the event of a collision):

	Swerve	Don't Swerve
Swerve	(keep car, keep car)	(lose car, win car)
Don't Swerve	(win car, lose car)	(lose car, lose car)

Give participants 10 minutes to come up with scenarios and then spend 5-10 minutes sharing. If you want, you can use scenario sharing following a break in order to re-focus the group.

ACTIVITY 2

(30-40 minutes)

THE LANGUAGE GAME

MATERIALS
- Large graph paper (1-inch squares)
- Markers

In the text, you learned about how the development of language among hominids has been modeled as a game. In this activity, you will play the role of people who have yet to agree on what words should be assigned to what objects.

Facilitator's note: This activity is to be done in pairs. Each person makes a word association list and then compares it to his or her partner's list. The pair's score for the round is equal to the number of agreements they share.

A

On a piece of paper, write the following nonsense words:

"oach"
"ehoobal"
"snuph"
"mootine"
"coffine"

Match each of the words above with one of the following objects:

morning
food
evening
flower
squirrel

Now, compare your list of associations with your partner's list, and give yourselves one point for each association that you agree on.

ACTIVITY 2

After figuring your score, each of you should independently reassign the "words" to the objects in an attempt to maximize your pair's score. Remember that you cannot communicate with each other unless you use the given words, so you cannot communicate to decide which words to use for which objects. After creating these new associations, come together again and calculate your new score.

Perform this reassigning exercise repeatedly in an attempt to reach the maximum score of 5.

1. How many rounds of reassignment does it take you and your partner to score a 5?
Answer: Answers will vary

Let the pairs play until they have all scored a 5. Then ask one representative from each pair to share the associations they settled on. Discuss how the associations were similar or different across pairs. If you have time and the group wants to, try the previous experiment again with players taking a new partner each round. See how long it takes to have the entire class come to agreement.

B

Play the same game as before with a new partner using the following lists of words and objects:

WORDS:
"blug"
"covlate"
"herthy"
"boox"
"celch"

OBJECTS:
morning
food
evening

1. How did the results of this game differ from those of the first game that you played?
Answer: Answers will vary.

ACTIVITY 2

C

Play the same game as before with a new partner using the following lists of words and objects:

WORDS:
"blug"
"celch"
"boox"

OBJECTS:
morning
food
evening
flower
squirrel

1. How did the results of this game differ from those of the previous games that you have played?
Answer: Answers will vary.

2. Is it easier to deal with different words that have the same meaning or different meanings assigned to the same word?
Answer: Answers will vary.

D

If time allows, try this activity:
This one can be a bit long, but interesting if done with the right groups of people.

Get a new partner and play the game again with the following long lists.

After each round, take a survey of the scores for each pair and graph the distribution on a "scores" vs. "number of pairs with that score" graph. Take the tally quickly and make the graphs on large sheets of graph paper while the pairs play the next round. Ideally, you will see the peak of the distribution move more and more toward the maximum score of "20" with each round.

ACTIVITY 2

WORDS:	OBJECTS:
“uslif”	boat
“istrith”	predator
“aplatain”	house
“witestent”	arm
“usrumpt”	kick
“evestigy”	bone
“trych”	edible plant
“strermas”	star
“akend”	name
“orm”	rain
“swousage”	daughter
“emivam”	son
“jasessa”	sun
“squupus”	moon
“stopoliso”	milk
“ondism”	water
“oflustoma”	bread
“reuride”	enemy
“crorp”	friend
“chaud”	dirt

If time allows, have the group play again with the above list, this time with a new partner each round. Is the difference in time taken to come to agreement for the whole class versus pairs significant?

For discussion:

1. What are the limitations of modeling language development in this fashion?
Answer: In real life there are many parts of speech, and players are not interacting in discrete rounds. Also, any particular “word” would most likely be heard in a particular contextual setting and language structure that would provide clues to its “meaning.” Furthermore, language can be passed down over generations, so very rarely would a group of people have to “start from scratch,” as they do in this game.

ACTIVITY 3

(30 - 40 minutes)

THE ULTIMATUM GAME

MATERIALS
- Large graph paper (1-inch squares)
- If possible and appropriate, some sort of incentives for the "winners" (e.g., food, coffee, math books)

The underlying assumption of classic game theory is that players act in a way that maximizes their gain and/or minimizes their loss. This is normally called "being rational." In real life, however, people are often not rational and behave in ways that they consider to be "fair," which is a more-nebulous concept. In this activity you will "measure" the fairness and rationality of your fellow participants by using the Ultimatum Game.

Rules

- There are two players, the offerer and the receiver.
- The offerer is given 100 imaginary dollars and is directed to split the money with the receiver.
- The offerer may choose how to split the $100. For example, he or she may keep $99 and offer $1 to the receiver; or the offered split might be 50/50; or the money can be divided in any number of other ways.
- The receiver, upon being offered some portion of the $100, gets to decide whether to accept or reject the offer. If the receiver accepts, then both players receive the portions they agreed to; if the receiver rejects the offer, neither player gets anything.

Split participants into pairs using some method of randomization—try not to have friends partnered with each other. Instruct them to keep track of their scores (what each player received) on paper for review after the first game. To make the games more exciting, you might get a selection of incentives (prizes) for people who "win" the various incarnations of the game.

ACTIVITY 3

A

Find a partner who is not a close friend. Flip a coin to decide who gets to be the offerer and who will be the receiver. The winner of the coin toss gets to choose.

Allow the offerer a minute or two to consider what his or her offer will be.

If you are the offerer, write down your offer on a piece of paper, but do not reveal it yet. Remember that the offer must be between $0 and $100.

Wait until everybody has written down an offer before having them show their offers to the receivers.

When the facilitator instructs you to, reveal your offer to the receiver.

If you are the receiver, take a minute or two to consider the offer and then write either "accept" or "reject" on the piece of paper.

If the offer is accepted, each player receives the amount agreed upon. If the offer is rejected, both players receive nothing.

Take a survey (anonymously—you can pass around a paper and have each pair write their "offerer score" and "receiver score"). Find an average score for offerers, an average score for receivers, and an average score for the pairs as a whole. Write these down for comparison with future results.

Discussion question: What do the average scores say about the overall concept of fairness within the group? (There are no right answers, but it should be interesting to see what people come up with.)

B

Play the Ultimatum again with a new partner. Make sure that if you were an offerer in the previous round you are paired with someone who was a receiver in the last round, so you can switch roles. Do not reveal to your partner what the outcome of the previous game was. Follow the same procedures.

When everyone has played, take another anonymous survey and find the averages mentioned above. Again, record these values for comparison with future results. Compare the results to the first-round results to see if greater knowledge of how the game works affected the overall group concept of fairness.

ACTIVITY 3

C

Find a new partner. Play eight consecutive rounds of the Ultimatum Game, alternating roles each round. Keep a running total of your scores. There is a prize for greatest total score.

After the group finishes play, take another anonymous survey to gather each pair's total score (no need to collect individual "offerer" or "receiver" information). Find an average total score and an average score "per round" (just divide the average total by 8). Compare the "per round" score to the scores from before.

Discussion questions: How did the playing of multiple rounds with the same person affect the "per round" score? How did the playing of the iterated Ultimatum Game compare to the previous "one-off" games?

D

Find a new partner and play the eight-round version of the game again. This time there is a prize for the lowest score.

Discussion question: How did the inverted incentive structure change the outcome of the game?

E

Before you play this game, decide as a group whether the prize should be for the highest score or the lowest score.

Play the one-shot Ultimatum Game once with each person in the room, keeping track of your score after each round. With each person you should play once as offerer and once as receiver. When you shift to a new partner, do not let them know what the outcomes of your previous rounds have been.

ACTIVITY 3

When you are finished, be ready to share your total score and average per-round score with the larger group.

There is no need for the survey to be anonymous after this game. Write down each person's name on the board, followed by their combined score and their average per-round score.

For discussion: Compare this version of the game to the one-shot and eight-round versions already played.

F

Before you play this game, decide as a group whether the prize should be for the highest score or the lowest score.

This will be the last game. Play again as you did in the game just completed, except this time, you are free to talk with your partners about one another's past results. You can ask your partners (opponents?) about their game history, but realize that they do not have to tell you the truth. Don't spend forever making a detailed chart of who is generous and who is stingy—there's not enough time to be thorough—but a little friendly gossip is encouraged!

Final discussion questions: What can you say about the group's concept of fairness? How does having a "reputation" affect this game?

ACTIVITY 4

[30-40 minutes]

In the last Ultimatum Game you played, you got a sense of the importance of reputation in iterated games. Many games can be played in iterated settings, including Prisoner's Dilemma. In this activity you will get a sense of how to go about formally examining and comparing strategies in iterated general Prisoner's Dilemma and the classic Rock, Paper, Scissors.

This is to be done in small groups. Be sure to take breaks to discuss results with the larger group, as necessary, depending on comprehension.

Recall that in a game of Prisoner's Dilemma (PD), each player has two possible strategies: cooperate (C) or defect (D). If both players cooperate, each receives the reward (R) payoff. If one cooperates and the other defects, the cooperator receives the sucker (S) payoff, and the defector receives the temptation to defect (T) payoff. If both players defect, each receives the punishment (P) payoff. In a standard PD, $T > R > P > S$.

1. Draw a payoff matrix for this game with player one on the left and player two on top. In each payoff cell, list player one's payoff first and player two's payoff second. Try not to look at the textbook while you do this.

Answer:

	Player 2		
Player 1		C	D
	C	(R, R)	(S, T)
	D	(T, S)	(P, P)

2. If this game is to be played only once, ask participants to explain why each player who is rational should choose to defect as long as $T > R > P > S$. Remember that players are not allowed to know what the other person chooses before making their choice.

Answer: Answers will vary, but should include the fact that choosing D ensures that, no matter what the other player does, a defector will never get the worst possible outcome, S. At the same time, defecting ensures that the opponent cannot get the best outcome, R.

ACTIVITY 4

3. Suppose that players are to play this game seven consecutive times. According to the one-shot PD logic, they should defect every time (which we'll call playing "AD," for "always defect"). Such a scenario can be represented by:

P1: DDDDDDD
P2: DDDDDDD

If this pattern is followed, find each player's score at the end of seven rounds in terms of TRSP. Express the result of a game extended to m rounds.
Answer: Both P1 and P2 will have the same score, 7P. After m rounds, their scores will be mP.

4. Iterated games allow for more variations of strategies than one-shot games. Might there be some strategy better than AD?"

Find the result of both players playing the strategy "always cooperate" (or AC) for m rounds. Compare this to the payoff for playing AD.
Answer:

P1: CCCCC...
P2: CCCCC...

Leads to these payoffs:

P1: RRRRR...
P2: RRRRR...

So, each player will receive mR after m rounds. Note that $mR \rightarrow mP$ because $R \rightarrow P$.

5. A conditional strategy is an iterated strategy in which a player changes his or her behavior based on what the other player has done in previous rounds. A classic conditional strategy is Tit-For-Tat (TFT), in which a player selects whatever strategy the opponent played in the last round.

Suppose that P2 is playing TFT. Complete the following game sequences to show how they would have played out in reaction to P1's strategy. In the first sequence, begin P2's play with a defect (D) and then pick up the TFT strategy; in the second sequence, begin with C. Compare the payoff results of each variant of TFT.

ACTIVITY 4

1st Sequence:
P1: DCDCDC
P2:

2nd Sequence:
P1: DCDCDC
P2:
Answer:
1st Sequence:
P1: DCDCDC
P2: DDCDCD

Payoffs:
P1: PSTSTS = 2T + P + 3S
P2: PTSTST = 3T + P + 2S

P2 wins because T → S.

2nd Sequence:
P1: DCDCDC
P2: CDCDCD

Payoffs:
P1: TSTSTS = 3T + 3S
P2: STSTST = 3T + 3S

Following the second sequence, P1 and P2 will play to a draw. Note that this would not be the case were there an odd number of rounds.

6. What would happen in a 7-round game if P1 plays AD and P2 plays D, then TFT?"
Answer:

P1: DDDDDDD
P2: DDDDDDD

Both players would receive 7P.

ACTIVITY 4

7. What would happen in a 7-round game if P1 plays AD and P2 plays C, then TFT?
Answer:

P1: DDDDDDD
P2: CDDDDDD

P1 gets T + 6P; P2 gets S + 6P. P1 would win.

8. Suppose that in a 7-round game P1 plays D, then TFT and P2 plays C, then TFT. What would happen?
Answer:

P1: DCDCDCD
P2: CDCDCDC

P1 gets 4T + 3S; P2 gets 4S + 3T. P1 would win.

9. What would happen if the above situation were extended to 8 rounds?
Answer: The result would be a draw, with each player receiving 4T + 4S.

10. What if both players were to play C, then TFT for 7 rounds?
Answer: Each player would receive 7R.

11. Compare the results of both players playing AD to both players playing C, then TFT for m rounds. What conclusions can you draw about the possible effectiveness of conditional strategies?
Answer: In the AD scenario, both players get mP. In the C, then TFT scenario, both players get mR. Because R → P, C, then TFT is a better strategy than AD—if—you happen to be playing someone who is playing the same strategy. So, in this case, the conditional strategy is superior to the "pure" strategy.

12. Express the payoffs to P1 and P2 in terms of TSRP if P1 plays AD and P2 plays C, then TFT for m rounds.
Answer:

P1: DDDDDD...
P2: CDDDDD...

P1 gets T + (m-1)P; P2 gets S + (m-1)P.

ACTIVITY 4

13. Suppose that P1 and P2 have the choice of playing either AD or C, then TFT (C, TFT). Create a payoff matrix that shows the payoffs for each player after m rounds.

Answer:

	Player 2		
Player 1		C, TFT	AD
	C, TFT	(mR, mR)	(S + (m-1)P, T + (m-1)P)
	AD	(T + (m-1)P, S + (m-1)P)	(mP, mP)

14. A Nash equilibrium occurs when neither player can do better by switching his or her strategy. In the one-shot PD, the Nash equilibrium was for each player to play AD. In the iterated PD, there are two Nash equilibria. The first is for both players to play AD. Explain why neither player can do better than playing AD.

Answer: If P1 plays TFT, then P2 should play AD because $T + (m-1)P > mR$. This is because $T > R$. If P1 plays AD, then P2 should play AD because $mP > S + (m-1)P$. This is because $P > S$. So, no matter what P1 does, P2 should play AD. Because the game is symmetric, the same logic goes for P1; therefore, both players should play AD.

15. To find the second Nash equilibrium, note that $T + (m-1)P > mR$ may not always be true! For what values of m will it be true that $mR > T + (m-1)P$?

Answer: Solve the inequality for m to find:

$$m > \frac{(T - P)}{(R - P)}$$

16. Use the results you just obtained to explain what the second Nash equilibrium is.

Answer: When the value of m makes $mR > T + (m-1)P$, then each player will have no incentive to play AD even though $T > R$. From P1's perspective, they will gain more by playing TFT than AD. P2's perspective is identical. P2 could play AD if the purpose was to see P1 lose, but P2 would also suffer in this situation because $T + (m-1)P < mR$. It is, therefore, in each player's best interest to play TFT.

ACTIVITY 4

17. Summarize your conclusions about the iterated PD in which the two optional strategies are AD and C, TFT.

Answer: If m is to be less than $\frac{(T-P)}{(R-P)}$, then it makes sense for each player to play AD. If m is to be more than $\frac{(T-P)}{(R-P)}$, then it makes sense for each player to play C, TFT. Everything, therefore, depends on the values of T, R, P, and S.

CONCLUSION

(30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Facilitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 9, Game Theory could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards?
Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: Harmonious Math. Workshop participants are expected to read the accompanying text for Unit 10, Harmonious Math before the next session

NOTES