MATHematics illuminated

PROGRAM: 8

Geometries Beyond Euclid

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Host: Dan Rockmore

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Time Code	Audio
00;15	OPENING CREDITS
00;40	HOST We live in a world — a reality —ruled by straight lines. Our streets, houses, cubicles, the shelves in our closets well, not my closet – virtually all of our space is parceled into rectilinear grids. Mathematicians were also ruled by straight lines - some would say imprisoned by them- for two thousand years. But what is a straight line? And when is a straight line not "straight"?
01;06	HOST Could it be… when you live in a curved world? These questions are at the heart of one of mathematics' most mind-bending revolutions... a revolution that created whole new geometries and whole new worlds...
01;21	**HOST (V.O.)** **The world is filled with an immense variety of shapes and structure. We use geometry to describe this physical space -- lines, points, angles and numbers characterize, organize and transform the shape of the world- even of space itself into coherent ideas.**
01;40	**HOST (V.O.)** **The word geometry comes from "geo" meaning earth and "meter" meaning to measure.** **And anyone who sees the pyramids knows that the ancients knew how to measure a straight line**
01;53	**HOST (V.O.)** **Much of early mathematics was all about geometry as well as simple computation. And it was largely developed for trade, agriculture, and building.**
02;01	**HOST (V.O.)** **But mathematics was also linked to religious observances, the motions of the planets and the construction of calendars.**
02;08	**HOST (V.O.)** **For the Greeks, mathematics was all about the tangible -- things we can see and touch, things that are real and measurable.**
02;16	**HOST (V.O.)** **Aristotle, in fact, considered even physics to be subservient to mathematics, believing that perfect motion had to take the form of the perfect geometric figures of straight lines and circles.**
02;26	**HOST (V.O.)** **The paradigm of this geometric outlook lies with the great Euclid. With one monumental work, Euclid set the stage for how we look at and measure the world geometrically. Except, there was a flaw - for**

	there are more than circles and lines in our universe...
02;48	HOST In his book, "the Elements", Euclid gathered all the theorems of his day into a framework of basic theory and proofs. In it, he laid down five postulates and invented a method by which we can prove geometry's most basic truths. And it all starts with two points and a line on a piece of paper… a two dimensional plane.
03;05	HOST The first three postulates are simple enough. The first states that a straight line segment can be drawn joining any two points. Just like that.
03;14	**HOST (V.O.)** **The second says, any straight line segment can be extended indefinitely in a straight line.**
03;19	**HOST (V.O.)** **The third states that given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center**
03;28	HOST The fourth states simply that all right angles are equal...
03;32	HOST and we need this so that we're sure the space we're working in is essentially the same everywhere.
03;37	HOST But the fifth is not so simple because it deals with the nature of parallel lines. It goes something like this:
03;43	**HOST (V.O.)** **Suppose a line segment intersects two straight lines in such a way that two interior angles on the same side add up to less than two right angles. Then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.**
04;00	HOST Now that sounds logical - in fact, so logical that mathematicians actually believed that it wasn't so much a postulate - that is an assertion about geometry that was independent of the other 4 postulates - but rather, it was a logical consequence of the other postulates. In other words, mathematicians seemed sure that all that we needed to know was in those first 4 postulates and that anything else could be derived from them. So, for two thousand years mathematicians tried to prove that fifth postulate from the other four, and failed. It became known as "the scandal of geometry". But why? Well, again implicit here is the idea of parallel lines. For what happens if those

	interior angles do add up to two right angles. Then the lines must never meet. Well maybe... The truth is, that there's more than one kind of geometry. You see, we don't live on a flat world - we live on a curved one. It's just that it's not obvious to us, at least not from where we're standing...
05;00	HOST If a man's walking on a surface then for all the he knows, he is on a plane. Locally, his world his flat. But if he's really on a curved surface...
05;10	**HOST (V.O.)** **...how does he know he's on a curved surface?** **From the point of view of just looking, he doesn't.**
05;14	HOST So for the Greeks, mathematics was largely about the geometry that they could see and measure in a rational way. So straight lines, right angles, parallel lines - those all seem rational and measurable, and seemed to be the ones they wanted for the flat world.
05;21	HOST And even though they knew the world was round, it wasn't something they could measure with a ruler, and so they ignored it. Logically, the five postulates made perfect sense... but from time to time, mathematicians challenged that fifth postulate, even as early as the fifth century, but it wasn't until the 19th century, when three mathematicians finally proved that there were indeed worlds out there and geometries that may seem impossible, but obey consistent logical laws. In other words, the fifth postulate was independent of the other four.
06;05	HOST One of those three mathematicians was Carl Friedrich Gauss...
06;09	**HOST (V.O.)** **Gauss, a professor at the University of Gottingen in Central Germany, was known even in his own time as a great mathematician. Napoleon's armies spared his town simply because he lived there. Today he is mentioned in the same breath with Archimedes and Isaac Newton.**
06;26	**HOST (V.O.)** **In 1822, Gauss was hired to conduct a survey of Hanover and make a map of the German countryside that would connect to an existing map of Denmark.**
06;35	**HOST (V.O.)** **Naturally, he was faced with the challenge of making a 2-dimensional map from 3-dimensional data...affected not just by changes in elevation but by the curvature of the earth. Because he was a**

	surveyor, he knew that, measured over great distances, a triangle's angles do not add up to 180 degrees...
06;54	**HOST (V.O.)** **But he also knew that the 180 degree rule was the equivalent of the fifth postulate. It is at this point that Gauss made one of those leaps of intuition that have often propelled mathematicians to whole new understandings of our universe.**
07;07	**HOST (V.O.)** **Gauss knew a triangle doesn't add up to 180 degrees in a curved world. This led him to wonder if space itself were curved?!!! What if he threw out the fifth postulate!**
07;17	**HOST (V.O.)** **Gauss realized was that there exists a whole different kind of geometry, one that moves beyond flat Euclidean 2-dimensional space into curved spaces. He was so unnerved by this realization that he didn't publish it. By not publishing Gauss muddied his legacy...**
07;32	**HOST (V.O.)** **...because within a few short years the Hungarian János Bolyai and the Russian Nikolai Lobachevsky both came up with a similar geometry - what they called "absolute geometry," because it was established without Euclid's fifth postulate!** **Still, the geometry they described was complicated and it took the mathematician Bernhard Riemann to think about this problem differently. He asked a more basic question - can I characterize a world where curvature is the same everywhere? And he did this using Gauss curvature.**
08;04	**HOST (V.O.)** **Instead of looking at triangles on a surface, we look at circles. If we use a piece of string of length r to trace out a circle on a flat surface, we know that we get a circle whose circumference is 2 Pi r. But if the surface is negatively curved, we get a number greater than 2 Pi r, and if it is positively curved, we get a number less than 2 Pi r.**
08;27	**HOST (V.O.)** **The discrepancy between the circumference of a circle on a curved surface and the circumference of a Euclidean circle of the same radius can be used to define curvature. With this notion of curvature, Riemann was able to classify surfaces of constant curvature whether it be positive, negative or zero.**
08;46	Dan Rockmore: I'm here with Daina Taimina, Adjunct Professor of Mathematics at Cornell University and an expert in helping people visualize non-Euclidian geometries. Daina, let's try to show people the distinctions between flat space and curved space.

09;00	Taimina: Here we have a flat paper, which can be a representation of how we usually include in geometry. Rockmore: Right, or a map, for example, right? Taimina: Yes. It can be -- this can be like something flat. Taimina: And here we are having a globe, which is a representation of the sphere. Rockmore: Right, a curved space.
9;16	Taimina: Yes. And, if our geometries would be the same, then we should have no problem of wrapping this paper around the ball like we are doing it was a present. Let's try. Rockmore: Let's give it a shot…another perfectly wrapped birthday present. So we've got – so we've got some places which are flattened. Taimina: Some places. Yeah, some places -- some places are flat, but then it gets too much and then this is not very nice. It doesn't look very nice. so can't really put it on. So -- And once we are opening…
09;49	Rockmore:: So -- so we're actually seeing something quite fundamental, flat space cannot go directly on to a curved space. There -- there will always be some problems…
09;58	Taimina: Yes, and those are -- those are the problems which mapmakers since ancient times have encountered, how to represent -- how to represent this curved space on the maps. So it means you have to decide what you are preserving, because there are different ways of making maps. You can preserve kind of like in angles or you can preserve like proportions between distances. Rockmore:: I can't do both at the same time. Rockmore:: There's always going to be some scrunching, just like we've seen here. Taimina: And that scrunching happens because this is different curvatures. Because our flat paper had zero curvature and this one is constant positive curvature. **Rockmore: this representation can't both be distance preserving and angle preserving. Is that right?** Taimina: Sure. Yes, because like flat maps that's a Euclidian Plain. And if we want to travel -- especially when we are doing air travel, so you have to take into account that our earth is more like a sphere,

10;51	Rockmore: so let's think about a potential travel plan. I'm -- I'm in Seattle and maybe I want to go to London, for example. So there's Seattle over there and there's London over there. Taimina: Yes, and in Euclidian Plain it would look like a straight line and -- and then like we see that -- that and we are used to that in Euclidian plains that the same -- same things as shortest distance would be the straight line segment between two points. Taimina: so, well, let's see what happens on a sphere.
11;08	Rockmore: And so now the shortest path I can imagine, think of that as a string and I want to think of using the least amount of string that's constrained to be on the globe, is that right? Taimina: Yes, it should be on a circle and that -- that is an arc of a great circle. Rockmore:: An arc of a great circle, which is like -- which is sort of like an equator that's at an arbitrary place. **Taimina: And now you see the difference between. Now you have two points, your Seattle and London, which is connected with two straight lines that is on a sphere.** Rockmore:: Ah. Right. Right. There's the short one and then there's this… Taimina: There is a short one. There is a long one. Rockmore:: There is this longer one. Taimina: So therefore we can see the shortest and straight is not the same on a sphere. It's another difference between two geometries. Rockmore: And so now we've seen two kinds of curvature. There's positively curved space. like the sphere. There's flat space like this tabletop and -- but, in fact, there are negatively curved spaces and they are very mysterious and interesting, like hyperbolic space. So let's take a look at one. Taimina: Yes.
12;00	Martin Steiner: I'm Martin Steiner, I'm at the University of North Carolina. I'm a Research Assistant Professor both in the department of Computer Science and Psychiatry, and this is sort of a weird association…what has Psychiatry to do with Computer Science? What we're doing here basically is neuroimaging.
12;17	Technician: **Ok, so, what I'm showing you here is a movie, of the average brain as a function of age, and so, what we're seeing is, now this is human brains, they're all healthy, and you can see that at the beginning, it's**

	age 30 and at the end, it's age 70 – they're all adults. And the idea of what I want to do is show what are the essential changes in the brain as a function of age.
12;42	Technician: And now what you can see is, I've taken all the actual…the images of the real subjects…these are real subjects, not averages, and put them together in this movie that shows the amazing amount of variability you see in the shape of the brain even between healthy subjects…
13;01	Technician: Now, what you're looking at here is this nonlinear shape space, and the idea is we want to compute the mean on this curved space rather than in Euclidean space…. And what we have to do is have a notion of centrality, that I'm depicting here, where this is the average that I'm going to compute, and its going to minimize the sum of the squared distances on this curve's manifold and that's how we define the average – using all of the brains that we've collected.
13;38	Steiner: What we're doing basically is developing tools for the analysis of medical images. Specifically we look at the brain mostly, we also look do other tools, but the brain is our major focus. So the brains are imaged on a scanner whether on an MRI scanner or a CT scanner, and the images are brought to us to evaluate them. So this is not something the radiologist does by simply looking at the images, but it's about extracting information that's more than just visually that can be seen in these images. So for example, a volume of a certain structure that might change over time and gives us information on whether that person may have alzheimer's or not…
14;19	Steiner: Since we're computer scientists, so we are actually programming mathematical algorithms that then read in these images, they extract certain values, certain measurements, and then give that measurement back either visually but also quantitatively in order to be analyzed by a statistician, or by a clinician. There's many things we do nowadays that have not been possible before. We started basically just looking at images of the brain and trying to understand where is white matter, where is grey matter, where is the fluid that nourishes the brain, the cerebral-spinal fluid, that was basically the earliest ways to look at the brain, and not even looking at this in any pathology – first trying to understand: can we evaluate the brain – these brain images with any kind of pattern-recognition algorithms that we were using back then maybe ten years ago.
15;10	Technician: So the reason that we need curved space is that the shape differences, and the shape changes that occur in the brain are occur locally, there's twisting and bending, there are small structures moving on one side of the image, while things are fixed on the other parts of the image and so

15;29	Technician: It also allows you to measure these local deformations, so changes in one part of the brain twisting, bending, things that you couldn't capture with the rotation or a translation, and finally, it allows you to- to, essentially a measuring stick in this curved space .
15;45	Steiner: And nowadays we are going more towards looking at the fine details of the brain. So nowadays we have algorithms that can take out of the brain certain structures that have a certain function, say the hippocampus, which is implicated in schizophrenia, in epilepsy, or in Alzheimer's disease to be different. We can measure the volume or even the shape of the hippocampus and compare that. That's something totally new that we wouldn't have been able to do ten years ago.
16;15	Technician: … so what I'm showing you is a 3D rendering at the top, but at the bottom, this is actually a slice that's taken out of this image of the average. And there you can see more deep structure… Steiner: So that looks kind of like a MRI slice. Technician: Exactly Steiner: But it's also not an actual MRI slice, but an average MRI slice. Technician: But an average…
16;33	Steiner: When one starts to analyze these images, one first tries the most simple tools, and that usually tools that are based in linear geometry, and you can only get so far with linear geometry because our brains are very different person to person, you often need to go to a nonlinear space. As soon as we want to use something like that, we need to have nonlinear geometry – we need to basically go from a level of mathematics that we started out from to a different level of mathematics, a higher level of mathematics, basically that we're using nowadays.
17;09	Steiner: Non-Euclidean geometry, or non-Euclidean analysis just gives us more detailed information, its simple not possible to do in a Euclidean way because the space that we operate in is a hyperbolic space, so its very easy if you just try to apply Euclidean geometry, you step out of the space of valid results.
17;32	Steiner: And for example if you look at shape analysis where we actually try to describe certain parts of the brain – let's say – the hippocampus, and we try to describe

	it, how it is shaped. All of that basically, whether it's now with a description of the interior which we use with a skeletal description, or the description of the interior which uses surface description – all both things we need non-EG. In the surface we go to spherical geometry, so we do everything on a sphere. For the medial description or skeletal description we need to go to a fully Rheimanian space, and do all our analysis in our Rheimanian space.
18;14	Steiner: so non-Euclidean geometry gives us new insight because it allows us to see differences, let's say, between one person the other, looking at the brain, not just in a linear fashion-So we really need to be able to do that, and it's kind of like a must for our research now-a-days.
18;33	Steiner: What I really love about my work, is really that, it's multifaceted.-The other thing of course, that's really inspiring is that we are helping people. That the goal of what we are doing is something that's towards medicine that then actually will be applicable.
18;50	HOST (V.O.) Flat surfaces and lines...
18;51	HOST (V.O.) ...spherical surfaces and lines... and two types of geometry to go with them. But the curved world, and a curved universe, don't stop with spherical geometry.
19;01	HOST (V.O.) What worried Gauss the most about throwing out Euclid's fifth postulate was that if he ended up with a geometry that admits triangles of greater than 180 degrees, it would be logical that there would also be a geometry that admitted triangles of less than 180 degrees.
19;17	HOST (V.O.) Bolyai and Lobachevsky came to the exact same conclusions, and we now call the geometry described by them and Gauss "hyperbolic geometry." Let's see if Daina can help see this world too...
19;30	Rockmore: Now those were some beautiful virtual hyperbolic planes, hyperbolic structures, but you're going to show us, in fact, that these are really tangible things as -- and beautiful.
19;40	Taimina: Yeah. You can -- we can have something tangible. Like, for example, we were saying something about the circle, where having the circle with a circumference in Euclidian plain. And then the circles, we are talking about the circles on a sphere. Now this is a crocheted version of so called sudosphere. Rockmore: A hyperbolic surface. Yeah. Right.
19;59	Taimina: this is what is characteristic for -- for a hyperbolic plain. It's like it's really connected with an exponential growth. Rockmore: That's right. So on a -- on a flat surface we -- we know that if you

	have a circle of radius R then the circumference is 2 pi R so that the circumference grows what we would say linearly, right? But here… Taimina: Yes. Yes. That grows exponentially.
20;20	Taimina: Now, let's go back to our old friend triangle. Rockmore: That's right, the triangle that we've been using to show the differences between all the geometries. Taimina: Yes. We know very -- very well. Now how do we get a triangle on this hyperbolic plain, which is essentially -- think about like a piece of paper. And now to get a straight line we can fold -- simply fold it like we would do… Rockmore: Right, one fold. Taimina:…it with a Euclidian plain. So we fold it -- we fold it so it's straight. Taimina: So then we can fold another one. Taimina: we are getting a triangle and we see these angles. Just visually we can see that these angles are much less. You know, like they are getting…the are becoming the same. And even more whats happening is that if we have a larger piece, ant then we are trying to fold it.
21;05	Rockmore: That's right. So now we are extending it. We're going to have longer and longer. Taimina: Yes. Think about them as they are extended longer and longer. So then we even get so called, which is called like so called ideal triangle. That means – Rockmore: Ideal, if we – if we had extended it to infinity it would be the ideal triangle. Taimina: Yes. If we extended all three sides to infinity we are getting ideal triangle; whereas the limit of the sum of interior angles goes to zero.
21;16	Rockmore: Right. Right. And so -- and so what about parallel lines in this world? Is there anything special about those?
21;30	Taimina: Well, yes, that is something. and this is something which was -- was Nicolai Bolyai and Lobachevsky when they were talking about that in -- in a space there is geometry where we can have a line. Here we have -- here we have a straight line and then we have -- we -- we can have more than one line, which is parallel to the given line. And let's see how it happens. That we can fold the line and we can see that this line -- Well, what do we call parallel? Now we have to think about it now. These lines won't intersect.
22;08	Rockmore: Not even out in infinity is the point.

	Taimina: No. No, that's the point. It's these lines won't intersect. So -- and these are non-intersecting lines. So that's what we call parallel lines. Rockmore: Non-intersecting lines. but the point being that they -- but that they all go through this point. Taimina: Yes. Exactly. Yes.
22;25	We can see that these ones, for example here there are three lines that go through the one line but none of them intersect. Rockmore: And they are all paralell to this where as in the plane we know that if you have one line and you take a point off the line then there is only one paralell line that goes through it.
22;40	Taimina: Exactly. Exactly. That's another difference between the geometries. That's a different behavior. Rockmore: Right
22;44	Taimina: And this is something -- this is something where our intuition just we don't have intuition about it. And that's about like the same thing about these other shapes, which is I have met as a scientist, which they are seeing something in the lab, which is like this. Rockmore: All of a sudden something clicks, is that right? Taimina: Yeah, something clicks and they say, oh! This looks like, you know, some of the mushrooms, or there are some… Rockmore: Sea anemone, for example. Taimina: Yes. Well, so you know, you see the shape and you have -- So then they say, well, we didn't know we have to use hyperbolic geometry. And that's -- that's when the visualization is a help.
23;14	Rockmore: Alright. So let's -- so let's summarize and sort of take our – our -- our tourist triangle on these three different geometries, right? So let's see. So the flat one there is our friendly triangle, a hundred eighty degrees no problem. Now he's going to wander off into -- into positively curved space on the sphere and fattens himself up, and more than a hundred and eighty degrees. And then he slouches off to hyperbolic space and he's thinned down, slimmed down to less than a hundred and eighty degrees.
23;38	Rockmore: Well Daina, thanks so much and, in particular, thanks for showing everybody how truly beautiful mathematics is and can be. Taimina: Thanks for hosting me. I'm glad to be here. Thank you.
23;52	HOST To get a picture of the hyperbolic plane on a larger scale requires some mind-

	bending ingenuity...
23;57	HOST One way to visualize this enigmatic space was developed by the great mathematician Henri Poincare. This is a Poincare disk - a mathematical model that's a "map" of the hyperbolic plane where the plane is compressed into a disk. Now remember, the hyperbolic plane has points that are infinitely far away. And in the mapping of the hyperbolic plane to the disk, points at infinity become the circumference.
24;23	HOST The effect of this is that long distances is the hyperbolic plane get squashed in Poincare disk. So as I move away from the center of the Poincare disk, approaching infinity, I'm actually moving exponentially far. Here's another way to look at it...
24;41	HOST Let’s make a beautiful pattern on the disk.-If we start with a Euclidean plane and tile it with a pattern...
24;48	HOST ...what we have is a repeating pattern that is reflected across straight lines.
24;53	HOST Now, let’s take another butterfly and build the analogous pattern with it on the Poincare disc, by reflecting that pattern across straight lines on the disc. See what we get?
25;17	HOST The model Poincare gave us is obviously useful to artists like Jos Leys, whose work we're looking at here. It's useful because it's what mathematicians call 'conformal' that is, you can measure the angles in the plane, they are the same in the disk because it's bounded by the circle, the design can be viewed in its entirety.
25;27	HOST Poincare's model offer a precise, aesthetically pleasing way to depict diminishing – and exponentially increasing – figures within a circle. Through its effect we see an illusion of negative curvature on this 2-dimensional surface.
25;41	HOST Leys' work, like that of the of 20th century Dutch artist M.C. Escher who inspired him, has produced hypnotically intricate patterns of images that seem otherworldly. And in fact, the hyperbolic plane lies outside the daily experience

	of the physical world. Or does it...?
25;58	HOST (V.O.) Daina Taimina's models of a surface with a constant negative curvature not only shows us what these surfaces look like... they also give us clues about how to find hyperbolic geometry in the natural world.
26;11	HOST (V.O.) It may have taken mathematicians 2000 years after Euclid to discover hyperbolic space, but nature discovered it millions of years ago —it's embodied in the structures of marine animals like sea-slugs, flatworms, and nudibranchs.
26;26	HOST When Gauss realized that mathematics could see beyond Euclid, it opened myriad possibilities. It's all about seeing ourselves like a bug on a curved world. You could say that, like the bug, we are prisoners of the space in which we live- unable to break free and observe the universe from the outside. But new geometries are giving mathematicians the ability, and the tools, to figure out things like the structure of space... and the shape of our brains or even the shape of the universe.
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