MATHematics illuminated

PARTICIPANT GUIDE

UNIT 8

UNIT 08

GEOMETRIES BEYOND EUCLID

PARTICIPANT GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

ACTIVITY 1

"A mathematician is a machine for turning coffee into theorems."
Apocryphal, although variously attributed to Alfred Rényi and Paul Erdös.

In Unit 8 of the Mathematics Illuminated textbook, we saw the connection between formal systems and geometry. We normally think of formal systems as logical rules and statements that can be made using those rules. Geometry is the study of space, angles, and shapes, but more than this it is a set of logical rules and statements that define how we expect space to "behave." In the text, we saw this in the example of Euclid's postulates, the fifth of which had different versions that led to different conceptions of space. The mini-activities for this unit will consist of exploring various non-geometric formal systems. All of the activities will use the following rules:

In this system there are three symbols: ▲, ●, and ■.

A theorem in this system will be a string of symbols, for example: ▲●●■, or ▲●■, or ▲■■▲■■.

An axiom is a statement from which other statements, theorems, can be proved using the rules. An axiom is different from a theorem, however, in that an axiom cannot be proved—it is postulated as true. There is one axiom in this system:

▲●

There are four rules:

I) If you have a string of symbols with ● as the last symbol, you can add a ■ onto the end.

Example: ▲● can turn into ▲●■.

II) If you have the string ▲x, with x representing any of the allowed symbols or a string of allowed symbols, then you may create ▲xx.

Examples: ▲● can turn into ▲●● and ▲●■ can turn into ▲●■●■.

ACTIVITY 1

III) If ●●● occurs in a string, it can be replaced with ■.

Example: ▲●●●● can turn into ▲●■ (or ▲■●). Note that this does not work in reverse, so you could not start with ▲●■ and get ▲●●●● using this rule.

IV) If ■■ occurs in a string, it can be dropped.

Example: ■■■ can turn into ■.

1. ▲●■ PUZZLE #1:

Start with the axiom and use the rules to derive ▲●■●■●■●■.
Give justification for each step you take.

2. ▲●■ PUZZLE #2:

Start with the axiom and use the rules to derive ▲■●■. Give justification for each step you take.

3. ▲●■ PUZZLE #3:

Give a few examples of strings that cannot be created starting from the axiom and using the rules. Be ready to explain why each one cannot be created in the ▲●■ system.

4. ▲●■ PUZZLE #4:

Start with the axiom and use the rules to derive the theorem: ▲■●●■. Give justification for each step you take.

5. ▲●■ PUZZLE #5:

How many theorems can be created if you start from the axiom and can use only four applications of the rules? How many of them are unique?

Hint 1: It might be useful to draw a tree, as in a factor tree or a decision tree. Start with the axiom at the top.

ACTIVITY 1

6. What is the minimum number of rules that must be applied before Rule IV is even an option? (In other words, what is the minimum number of steps in a derivation that can be taken before Rule IV becomes applicable?)

7. How does the study of the ▲●■ system relate to geometry? (think about proofs) What does it have to do with mathematics?

IF TIME ALLOWS:

8. Is ▲■ a theorem in the ▲●■ system? Why or why not?

ACTIVITY 2

MATERIALS

- Tennis ball
- Soccer ball
- Large beach ball
- Football
- String
- Compass
- Pushpins (for tennis ball)
- Graph paper
- Colored pencils
- Overhead markers

To be done in a small group of three or four.

In traditional (flat, Euclidean) geometry, the circumference of a circle is proportional to twice its radius. The constant of proportionality is normally referred to as π, which is equal to 3.141596.... If the world or universe is Euclidean, we should be able to go to any place and draw a circle of any radius and always get the same value for π by dividing the measured circumference by twice the measured radius. This same relationship does not necessarily hold for geometry on curved surfaces. In this activity you will explore what happens to π for curved surfaces.

In your groups, measure how the circumference of a circle drawn on the surface of a curved surface relates to its radius.

Suggested steps:

- For each ball, cut a length of string equal to its circumference. Mark the string in 1-cm increments. Anchor the string using tape or a pushpin (if using the tennis ball) somewhere on the ball; we'll call this spot the "north pole."

- Use the marker in conjunction with the string to draw circles, analogous to "lines of latitude," at each radial increment of 1 centimeter.

- Use another piece of string to measure the length (circumference) of each circle.

ACTIVITY 2

- Divide the circumference of each circle by twice its radius to find "π."

- Make a graph of the radius vs. "π" for all the circles on each ball.

- For the football, you should create two graphs, one for circles centered at one of the "pointy" ends and one for circles centered in the middle, flatter region of the ball.

- When you are finished, you should have five graphs: tennis ball, soccer ball, large beach ball, pointy end of the football, and the middle of the football.

After you have made the graphs, answer the following questions and be ready to discuss your results with the large group.

1. Is "π" a constant for the surfaces you measured? Explain.

2. How does the rate of change of "π" correlate with the curvature of the surface?

3. Compare the values of "π" found for small radii on the five graphs.
What can be said about the local curvature of each surface?

4. Would you say that this experiment measures an intrinsic or extrinsic property of the surfaces? Explain.

5. All of the surfaces measured in this experiment are positively curved. How would you expect the results to differ if a similar experiment were performed on surfaces having negative curvature, such as a saddle or a "wrinkly" cloth? Sketch a possible graph of the radius vs. "π" for a concentric set of circles on a hypothetical surface of negative curvature.

Hint 1: It might help to refer to Unit 8 of the text.

ACTIVITY 3

In the text you read about the 17th/18th century Jesuit priest, Girolamo Saccheri, who attempted to prove the necessity of Euclid's fifth postulate by examining the summit angles of quadrilaterals with right base angles. He studied cases in which the sum of the summit angles could be more than or less than the sum of two right angles and attempted to show that these ideas were absurd. He failed of course because, as it turns out, there are many versions of Euclid's fifth postulate that all lead to consistent geometries. In this activity, you will explore the connections between Saccheri's quadrilaterals and the angle sum of a triangle. This leads to the idea that triangles drawn on a positively curved surface (one in which there are no parallel lines possible through a given point) have an angle sum greater than 180 degrees. Conversely, triangles drawn on a negatively curved surface (one in which there are many parallel lines possible through a given point) have an angle sum less than 180 degrees.

A

Assume that the "no parallels" version of the fifth postulate holds.

Let the "excess" of a triangle be the difference between its angle sum and 180 degrees. Let the "excess" of a quadrilateral be the difference between its angle sum and 360 degrees.

1. First, show that congruent triangles have the same excess.

2. Now show that the excess of quadrilateral PQRS is equal to the sum of the excesses of triangles PSR and PQR.

3. For the following figure, show that the excess of triangle ABC is equal to the excess of quadrilateral BCHG.

4. Let D be an arbitrary point on side $\overline{BC}$ of triangle ABC. Show that the excess of ABC is equal to the sums of the excesses of ABD and ACD.

5. For any triangle, show that there must exist another triangle whose excess is, at most, half that of the given triangle.

6. Finally, show that there must exist a triangle with an excess as small as we like. What is the angle sum of such a triangle?

7. How does this line of reasoning tie together triangle sums with Saccheri's quadrilaterals?

ACTIVITY 3

B

Assume that the "many parallels" version of the fifth postulate holds.

Let the "defect" of a triangle be the difference between 180 degrees and its angle sum. Let the "defect" of a quadrilateral be the difference between 360 degrees and its angle sum.

1. Follow a line of reasoning similar to that before to show that in this geometry there must exist a triangle whose defect is as small as we like. Then find the angle sum of this triangle.

Hint 1: Replace "excess" with "defect" and proceed.

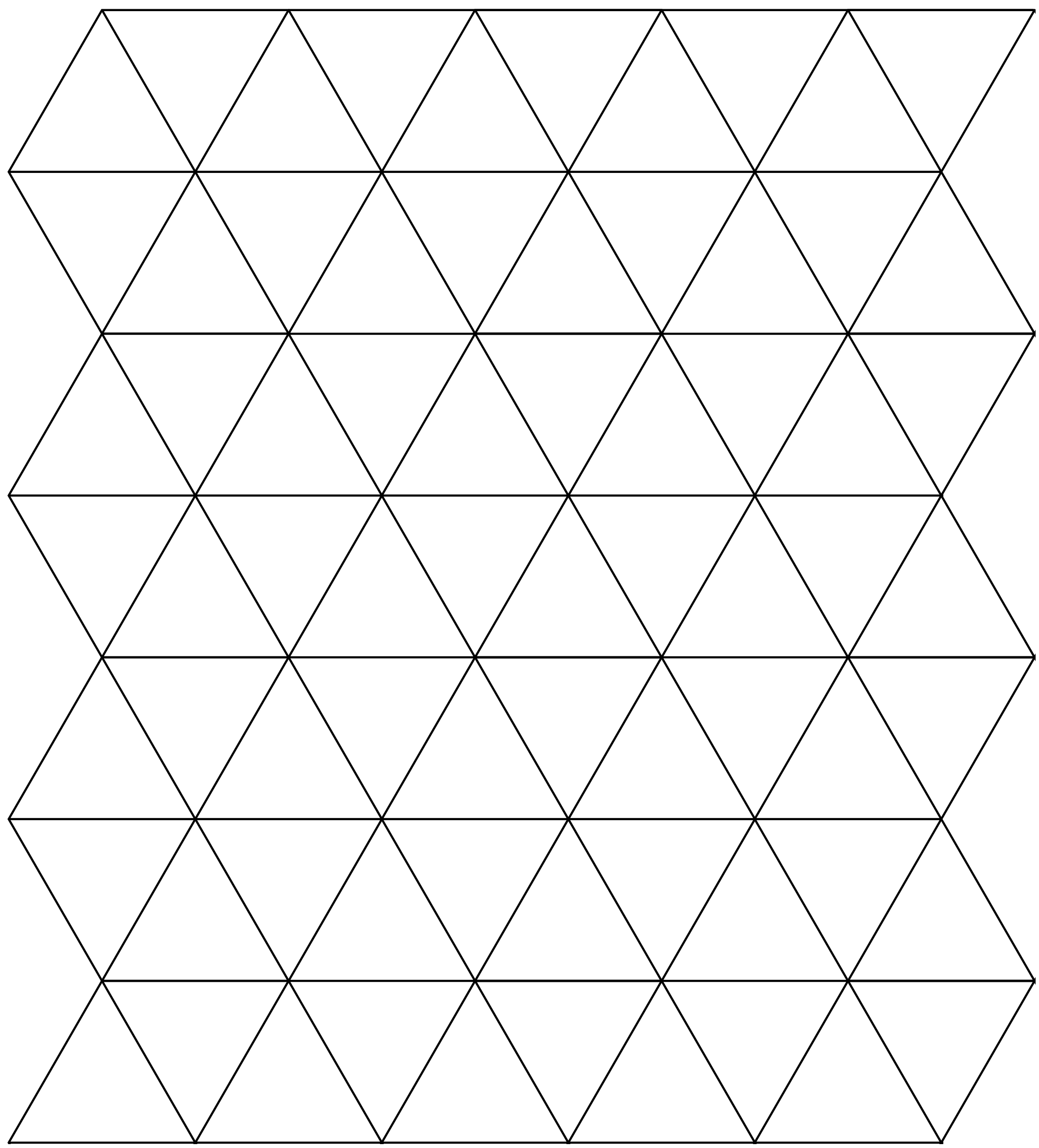
2. Discuss the relationship between Saccheri quadrilaterals and triangle sums on positively and negatively curved surfaces.

ACTIVITY 4

MATERIALS

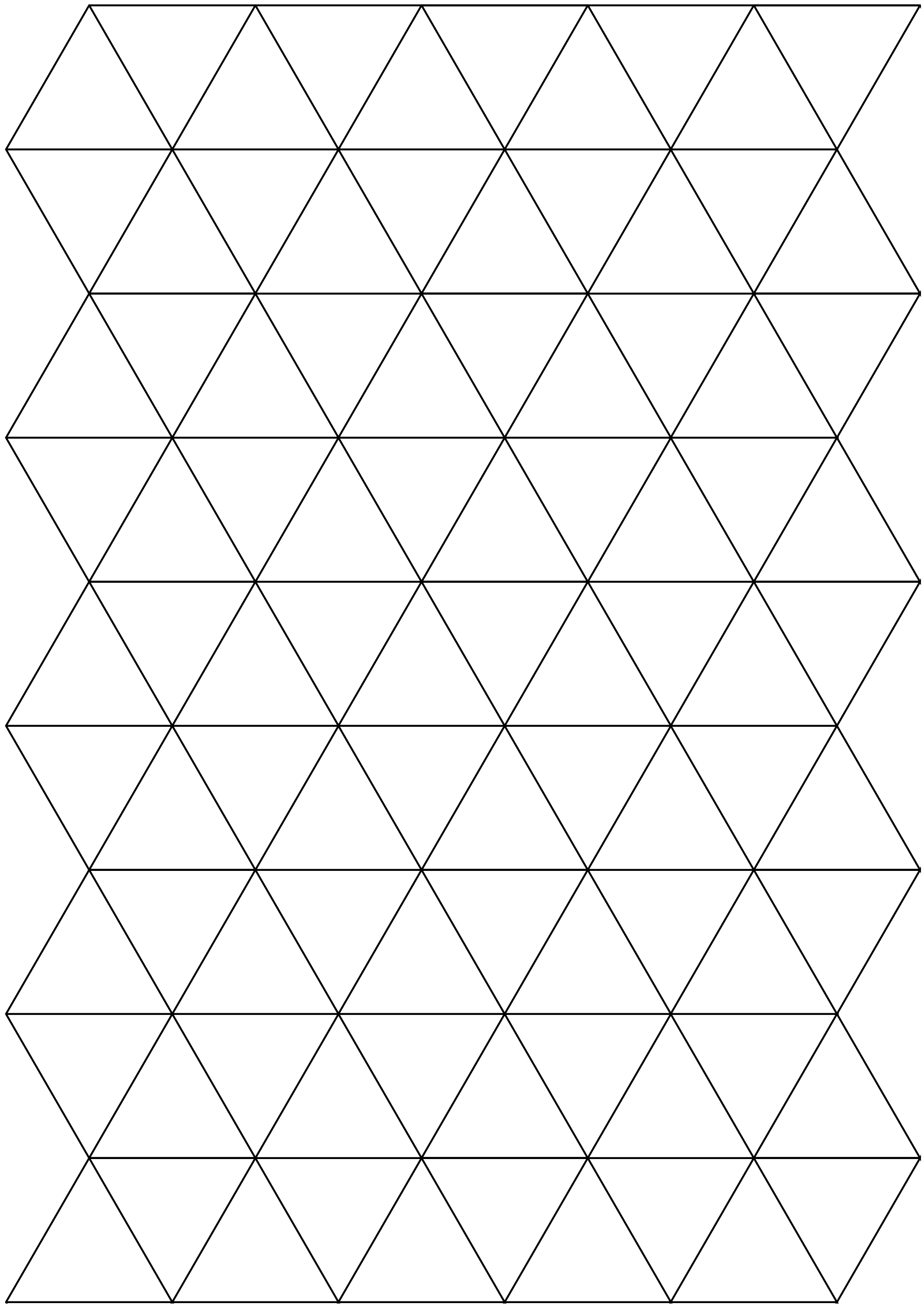
- Equilateral triangular graph paper (provided)—about five sheets per table
- Scissors
- Tape

Equilateral Triangular Graph Paper Worksheet



ACTIVITY 4

Equilateral Triangular Graph Paper Worksheet Continued



ACTIVITY 4

In this activity you will build models of non-Euclidean geometries using familiar equilateral triangles.

Using a sheet of the triangular graph paper, draw an approximate Euclidean disk in this tiling, which we will call a "radius-n Euclidean comb-disk." To form it, let the radius-0 Euclidean comb-disk be a single vertex in the tiling. Now we can expand a radius-(n–1) Euclidean comb-disk to a radius-n Euclidean comb-disk by attaching all the triangles in our lattice that have a vertex in the radius-(n–1) Euclidean comb-disk.

1. On a sheet of triangular graph paper, draw Euclidean comb-disks of radius 0, 1, 2, 3, and 4 centered at a fixed vertex.

2. The circumference of a Euclidean comb-disk is the number of edges on the disk boundary. Graph the circumference as a function of the radius up to radius 4.

3. Find a formula for the circumference of a Euclidean comb-disk as a function of its radius.

Now we are going generalize our Euclidean comb-disk. Let the spherical comb-disk be built from the same equilateral triangles as the Euclidean comb-disk, by taping the triangles together such that there are five triangles at each vertex (rather than six, as in the Euclidean case). Cut out the triangles and start taping them together!

4. Argue that intrinsically this space is flat everywhere, except at the vertices.

5. What is the circumference of a small unit circle at a vertex?

6. Does this correspond to positive or negative curvature?

7. Form the comb-sphere by taping until you can tape no more.
Can you describe the resulting shape?

8. How much total curvature did you find?

ACTIVITY 4

9. The Gauss-Bonnet theorem (widely considered the most beautiful theorem in mathematics) tells us that the total curvature of a closed surface is $2 \times \pi \times$ (Euler Characteristic). Is this theorem consistent with what we found with the comb-sphere?

10. Graph the circumference of the spherical comb-disk as a function of the radius up to radius 3.

Let the hyperbolic comb-disk be built from the same equilateral triangles as the Euclidean and spherical comb-disks, but this time tape triangles together so that there are seven triangles at each vertex. Cut out the triangles and start taping them together!

11. Notice that this is also flat everywhere except at the vertices. What is the circumference of a small unit circle at a vertex?

12. Does this correspond to positive or negative curvature?

13. Graph the circumference of the hyperbolic comb-disk as a function of the radius up to radius 3.

14. (Challenging) We are building the hyperbolic plane; hence, as we saw in the video, this comb-disk of radius r should have loads of room. Show that the circumference of the hyperbolic comb-disk satisfies $C(n)>2^n$.

15. Using the estimate from the previous problem, at least how much larger is the circumference of a hyperbolic comb-disk of radius 100 than a Euclidean comb-disk of radius 100?

16. (More Challenging) The usual hyperbolic plane cannot be embedded in the Euclidean-like space we live in (locally!). Show that for some *n* you cannot build a hyperbolic comb-disk of radius *n*.

17. Here is a so-called "open problem," one that as of this printing has yet to be solved: Find the largest-radius hyperbolic comb-disk that CAN be embedded in Euclidean space. (If you tried it, could you build the radius-2 hyperbolic comb-disk? How about the one of radius 3?)

CONCLUSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

DISCUSSION

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 8, Geometries Beyond Euclid could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no “frontloading?”

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: Game Theory. Workshop participants are expected to read the accompanying text for Unit 9, Game Theory before the next session.

NOTES