MATHematics illuminated

FACILITATOR GUIDE

UNIT 8

UNIT 08

GEOMETRIES BEYOND EUCLID

FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

ACTIVITY 1

(45 minutes)

Facilitator's note: feel free to have participants work individually or in groups of up to three. Discussion breaks are not explicitly called for in this activity, but please use your discretion to make sure that participants are understanding.

"A mathematician is a machine for turning coffee into theorems."
Apocryphal, although variously attributed to Alfred Rényi and Paul Erdös.

In Unit 8 of the Mathematics Illuminated textbook, we saw the connection between formal systems and geometry. We normally think of formal systems as logical rules and statements that can be made using those rules. Geometry is the study of space, angles, and shapes, but more than this it is a set of logical rules and statements that define how we expect space to "behave." In the text, we saw this in the example of Euclid's postulates, the fifth of which had different versions that led to different conceptions of space. The mini-activities for this unit will consist of exploring various non-geometric formal systems. All of the activities will use the following rules:

In this system there are three symbols: ▲, ●, and ■.

A theorem in this system will be a string of symbols, for example: ▲●●■, or ▲●■, or ▲■■▲■■.

An axiom is a statement from which other statements, theorems, can be proved using the rules. An axiom is different from a theorem, however, in that an axiom cannot be proved—it is postulated as true. There is one axiom in this system:

▲●

There are four rules:

I) If you have a string of symbols with ● as the last symbol, you can add a ■ onto the end.

Example: ▲● can turn into ▲●■.

ACTIVITY 1

II) If you have the string ▲x, with x representing any of the allowed symbols or a string of allowed symbols, then you may create ▲xx.

Examples: ▲● can turn into ▲●● and ▲●■ can turn into ▲●■●■.

III) If ●●● occurs in a string, it can be replaced with ■.

Example: ▲●●●● can turn into ▲●■ (or ▲■●). Note that this does not work in reverse, so you could not start with ▲●■ and get ▲●●●● using this rule.

IV) If ■■ occurs in a string, it can be dropped.

Example: ■■■ can turn into ■.

1. ▲●■ PUZZLE #1:

Start with the axiom and use the rules to derive ▲●■●■●■●■.
Give justification for each step you take.
Answer: Start with ▲●, and use Rule I to get ▲●■; use Rule II to get ▲●■●■; use Rule II again to get ▲●■●■●■●■.

2. ▲●■ PUZZLE #2:

Start with the axiom and use the rules to derive ▲■●■. Give justification for each step you take.
Answer: Start with ▲●, and use Rule II to get ▲●●; use Rule II again to get ▲●●●●; use Rule I to get ▲●●●●■; use Rule III to get ▲■●■.

3. ▲●■ PUZZLE #3:

Give a few examples of strings that cannot be created starting from the axiom and using the rules. Be ready to explain why each one cannot be created in the ▲●■ system.
Answer: Answers will vary, but be sure that the participants have valid reasons why their strings are not theorems of the system. Examples: ■●■ cannot be created because it does not start with ▲, and there is no rule that allows for the creation or destruction of a ▲. The string ▲▲▲▲ cannot be reached for the same reasons. Be careful with statements about strings such as ▲■, which may or may not be possible and is difficult to prove either way.

ACTIVITY 1

4. ▲●■ PUZZLE #4:

Start with the axiom and use the rules to derive the theorem: ▲■●●■. Give justification for each step you take.

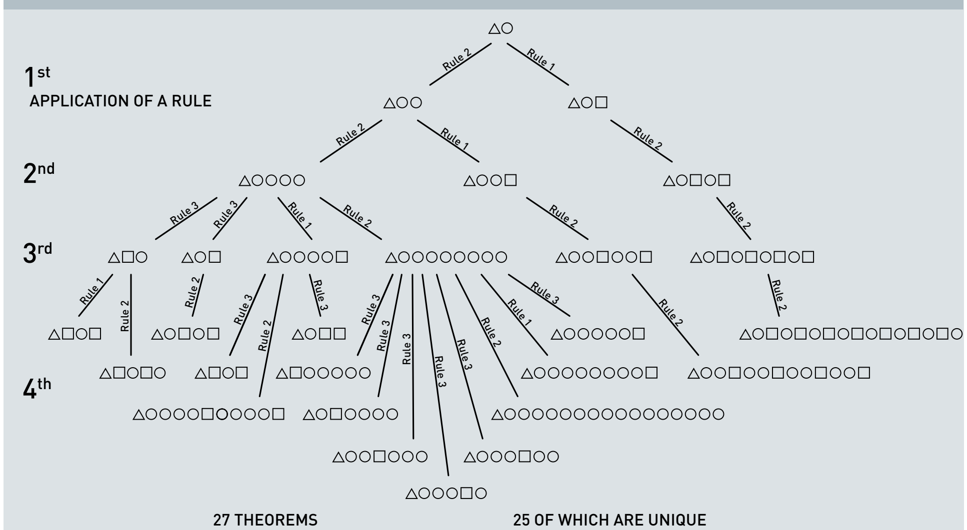
Answer: Start with the axiom, ▲●, and use Rule II to get ▲●●, use Rule II again to get ▲●●●●; use Rule I to get ▲●●●●■, use Rule III to get ▲■●■; use Rule II to get ▲■●■■●■; use Rule IV to get ▲■●●■.

5. ▲●■ PUZZLE #5:

How many theorems can be created if you start from the axiom and can use only four applications of the rules? How many of them are unique?

Hint 1: It might be useful to draw a tree, as in a factor tree or a decision tree. Start with the axiom at the top.

Answer:



Answer: There are 27 possible theorems, 25 of which are unique.

6. What is the minimum number of rules that must be applied before Rule IV is even an option? (In other words, what is the minimum number of steps in a derivation that can be taken before Rule IV becomes applicable?)

Answer: Rule IV requires two consecutive ■'s. A quick glance at the tree shows that this cannot happen before the application of the fourth round of rule application, so Rule IV could potentially be of use in the fifth round of application, or identically, the fifth step of a derivation.

ACTIVITY 1

7. How does the study of the ▲●■ system relate to geometry? (think about proofs) What does it have to do with mathematics?

Answer: Answers will vary, but traditional geometry classes are centered around proving theorems about different shapes using a given set of rules as to what is allowed. All of the geometry in Euclid's Elements is based upon five postulates. Mathematics is thought to be a vast construction of theorems built upon a foundation of axioms or postulates that are "self-evident." The axiom and rules of the ▲●■ system are contrived, but they serve to show how the fundamental act of doing mathematics works.

IF TIME ALLOWS:

8. Is ▲■ a theorem in the ▲●■ system? Why or why not?
Answer: ▲■ is impossible to create in this system. It is difficult to prove this formally, but a bit of exploration and trial-and-error should convince the participants that this is the case.

ACTIVITY 2

(30 minutes)

Facilitator's note: it will be beneficial to try this one out before the session so that you can better help participants. It is not structured in terms of times because of its rather open-ended nature. Be sure to circulate often, checking to make sure groups understand the directions.

MATERIALS

- Tennis ball
- Soccer ball
- Large beach ball
- Football
- String
- Compass
- Pushpins (for tennis ball)
- Graph paper
- Colored pencils
- Overhead markers

To be done in a small group of three or four.

In traditional (flat, Euclidean) geometry, the circumference of a circle is proportional to twice its radius. The constant of proportionality is normally referred to as π, which is equal to 3.141596.... If the world or universe is Euclidean, we should be able to go to any place and draw a circle of any radius and always get the same value for π by dividing the measured circumference by twice the measured radius. This same relationship does not necessarily hold for geometry on curved surfaces. In this activity you will explore what happens to π for curved surfaces.

In your groups, measure how the circumference of a circle drawn on the surface of a curved surface relates to its radius.
Each small group can measure one of the balls and then trade with another group when they are finished. The order in which the balls are measured does not matter.

ACTIVITY 2

Suggested steps:

- For each ball, cut a length of string equal to its circumference. Mark the string in 1-cm increments. Anchor the string using tape or a pushpin (if using the tennis ball) somewhere on the ball; we'll call this spot the "north pole."
- Use the marker in conjunction with the string to draw circles, analogous to "lines of latitude," at each radial increment of 1 centimeter.
- Use another piece of string to measure the length (circumference) of each circle.
- Divide the circumference of each circle by twice its radius to find "π."
- Make a graph of the radius vs. "π" for all the circles on each ball.
- For the football, you should create two graphs, one for circles centered at one of the "pointy" ends and one for circles centered in the middle, flatter region of the ball.
- When you are finished, you should have five graphs: tennis ball, soccer ball, large beach ball, pointy end of the football, and the middle of the football.

After you have made the graphs, answer the following questions and be ready to discuss your results with the large group.

1. Is "π" a constant for the surfaces you measured? Explain.
Answer: Answers will vary. Participants should find that "π" decreases as the radius increases, although at different rates for each surface. Thus, it is not constant, both for each surface and for the group of surfaces as a whole.

2. How does the rate of change of "π" correlate with the curvature of the surface?
Answer: Answers will vary, but in general, the more curved a surface, the faster "π" decreases as the radius increases.

3. Compare the values of "π" found for small radii on the five graphs.
What can be said about the local curvature of each surface?
Answer: The values of "π" for small radii on each surface should be relatively close to 3.1415.... This implies that the local curvature is close to flat.

ACTIVITY 2

4. Would you say that this experiment measures an intrinsic or extrinsic property of the surfaces? Explain.

Answer: Answers will vary but should be along the lines of: "This involves intrinsic measurement, because it could be done by an ant on any of the surfaces. No extrinsic views are required."

5. All of the surfaces measured in this experiment are positively curved. How would you expect the results to differ if a similar experiment were performed on surfaces having negative curvature, such as a saddle or a "wrinkly" cloth? Sketch a possible graph of the radius vs. "π" for a concentric set of circles on a hypothetical surface of negative curvature.

Hint 1: It might help to refer to Unit 8 of the text.

Answer: "π" increases as the radius increases on negatively curved surfaces, and the graph should reflect this.

Convene the large group to discuss the results.

ACTIVITY 3

(30 - 40 minutes)

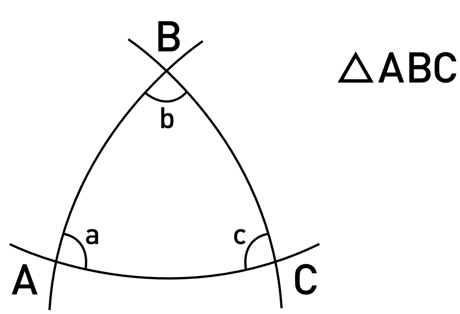
In the text you read about the 17th/18th century Jesuit priest, Girolamo Saccheri, who attempted to prove the necessity of Euclid's fifth postulate by examining the summit angles of quadrilaterals with right base angles. He studied cases in which the sum of the summit angles could be more than or less than the sum of two right angles and attempted to show that these ideas were absurd. He failed of course because, as it turns out, there are many versions of Euclid's fifth postulate that all lead to consistent geometries. In this activity, you will explore the connections between Saccheri's quadrilaterals and the angle sum of a triangle. This leads to the idea that triangles drawn on a positively curved surface (one in which there are no parallel lines possible through a given point) have an angle sum greater than 180 degrees. Conversely, triangles drawn on a negatively curved surface (one in which there are many parallel lines possible through a given point) have an angle sum less than 180 degrees.

A (15-25 minutes)

Assume that the "no parallels" version of the fifth postulate holds.

Let the "excess" of a triangle be the difference between its angle sum and 180 degrees. Let the "excess" of a quadrilateral be the difference between its angle sum and 360 degrees.

1. First, show that congruent triangles have the same excess.
If participants need a "jumpstart," you can draw the following picture on the board:

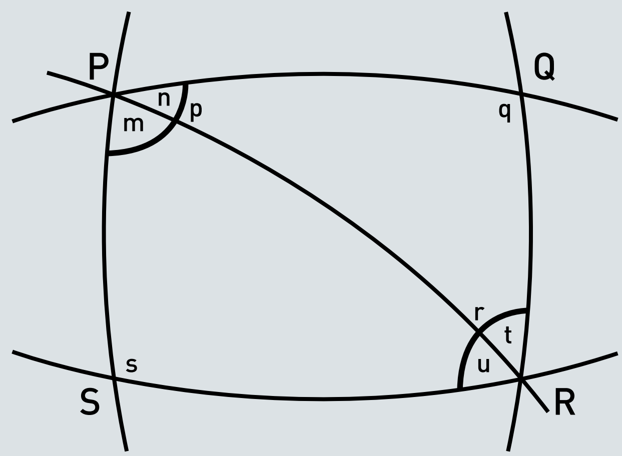


Answer: The excess of triangle ABC is 180 – (a + b + c). Any triangle congruent to ABC will have angles equivalent to a, b, and c; therefore, the excess of any congruent triangle will be equal to 180 – (a + b + c) also.

ACTIVITY 3

2. Now show that the excess of quadrilateral PQRS is equal to the sum of the excesses of triangles PSR and PQR.

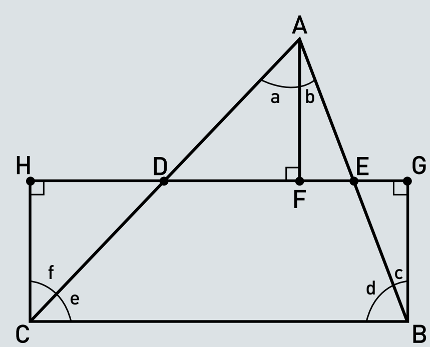
Answer:



Answer: The excess of PQRS is 360 – (p + q + r + s). The excess of PSR is 180 – (s + m + u). The excess of PQR is 180 – (q + n + t). The sum of the excesses of PSR and PQR is 360 – (m + n + q + u + t + s). Recognizing that m + n = p and u + t = r leads to: 360 – (p + q + r + s). Q.E.D.

3. For the following figure, show that the excess of triangle ABC is equal to the excess of quadrilateral BCHG.

Answer:



D IS THE MIDPOINT OF $\overline{CA}$

E IS THE MIDPOINT OF $\overline{AB}$

$\overline{AF} = \overline{CH} = \overline{BG}$

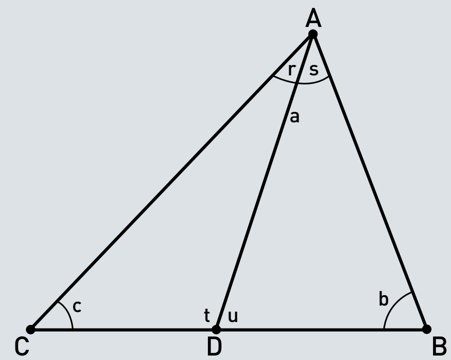
$\triangle AFD \cong \triangle CHD$ and $\triangle AEF \cong \triangle BGE$

Answer: The excess of ABC is 180 – (a + b + d + e). The excess of BCHG is 360 – 90 – 90 – (f + e + d + c) = 180 – (f + e + d + c). To complete this, we need to show that (a + b) = (f + c). This follows directly from the fact that triangles AFD and CHD are congruent, as are triangles AFE and BGE. Because the excesses of congruent triangles are equal to one another, we can show that b = c and a = f.

ACTIVITY 3

4. Let D be an arbitrary point on side $\overline{BC}$ of triangle ABC. Show that the excess of ABC is equal to the sums of the excesses of ABD and ACD.

Answer:



Answer: The excess of ABC is 180 – (a + b + c). The excess of ACD is 180 – (c + r + t). The excess of ABD is 180 – (b + s + u). Adding the excesses of ACD and ABD gives: 360 – (b + c + r + s + t + u). We know that t + u = 180 and r + s = a, so substituting these in gives: 180 – (a + b + c). Q.E.D.

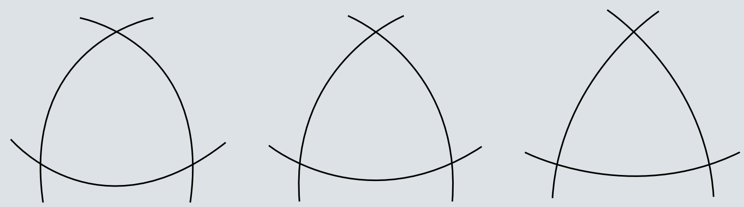
5. For any triangle, show that there must exist another triangle whose excess is, at most, half that of the given triangle.

Answer: From the above result, it is clear to see that no matter where D is on $\overline{BC}$, it will divide the arbitrary triangle ABC into two subtriangles whose excesses sum to the excess of ABC. Unless ABC is isosceles and D is the midpoint of $\overline{BC}$, the excess of one of the subtriangles will be less than half the excess of ABC. If ABC is isosceles and D is the midpoint of $\overline{BC}$, then the excess of each subtriangle will be exactly half of the excess of ABC. Therefore, for any triangle ABC, there will be a subtriangle with an excess of, at most, half that of ABC.

ACTIVITY 3

6. Finally, show that there must exist a triangle with an excess as small as we like. What is the angle sum of such a triangle?

Answer:



Answer: The above image shows that the curvature of the geodesics that make up the sides of a triangle correlates with the excess of that triangle. As the sides become flatter, the angles a, b, and c come closer and closer to summing to 180 degrees. We can have geodesics as close to flat as we would like, so we can have angle sums as close as we like to 180 degrees. Note that the angle sum can never equal 180 degrees, however, because we are not allowed to have any parallel lines. Therefore, we can have a triangle whose excess gets close to, but never reaches, zero. The angle sum of this triangle is arbitrarily more than 180 degrees.

Convene the large group and discuss the results. (5 minutes)

7. How does this line of reasoning tie together triangle sums with Saccheri's quadrilaterals?

Answer: In the third diagram, quadrilateral BCGH is an inverted Saccheri quadrilateral in which f + e and d + c are "excessive" summit angles. The fact that the excesses of BCGH and ABC are equal ties together triangle sum excesses and quadrilateral sum excesses.

ACTIVITY 3

B (10 minutes)

Assume that the "many parallels" version of the fifth postulate holds.

Let the "defect" of a triangle be the difference between 180 degrees and its angle sum. Let the "defect" of a quadrilateral be the difference between 360 degrees and its angle sum.

1. Follow a line of reasoning similar to that before to show that in this geometry there must exist a triangle whose defect is as small as we like. Then find the angle sum of this triangle.

Hint 1: Replace "excess" with "defect" and proceed.
Answer: Answers will vary but should be logical. They do not need to follow the line of reasoning used before but should make sense and be reasonable to others in the large group.

2. Discuss the relationship between Saccheri quadrilaterals and triangle sums on positively and negatively curved surfaces.
Answer: Answers will vary; have the participants discuss this. Steer them toward these realities: summit angles less than 180 degrees correspond to spaces in which triangle sums are less than 180 degrees, and summit angles more than 180 degrees correspond to spaces in which triangle sums are more than 180 degrees.

Convene the large group and discuss the results. (5 minutes)

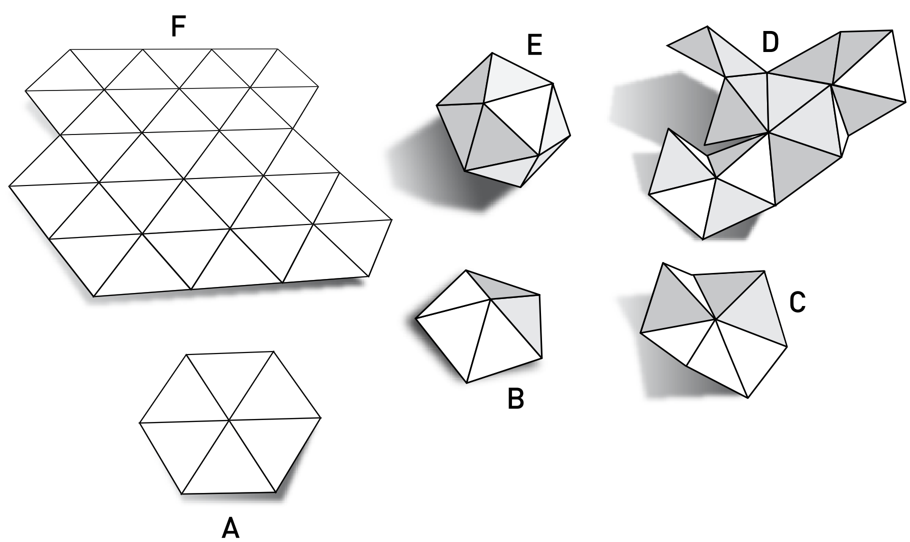
ACTIVITY 4

(50 minutes)

MATERIALS

- Equilateral triangular graph paper (provided)—about five sheets per table
- Scissors
- Tape

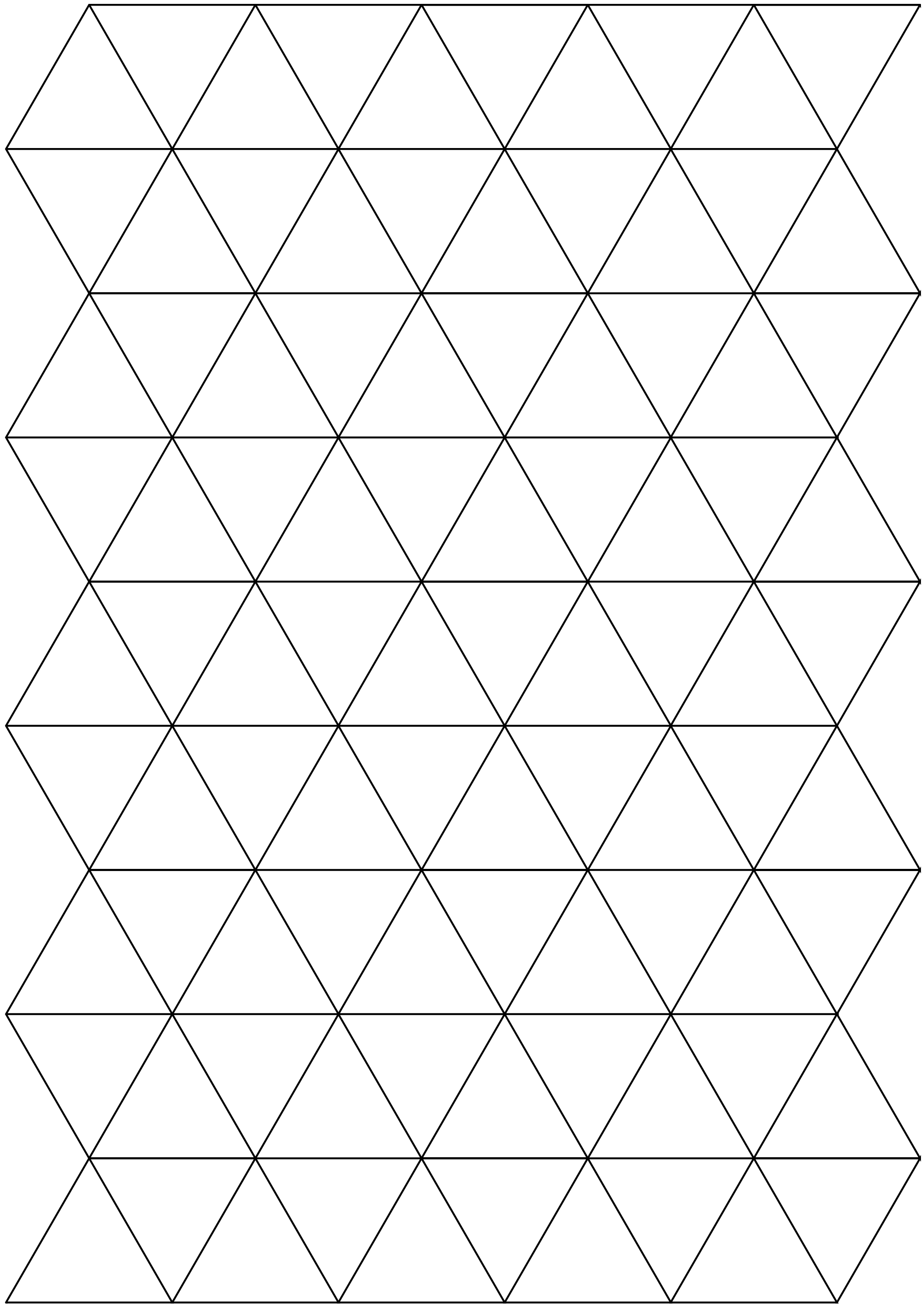
Facilitator's note: This activity is appropriate for more-experienced and/or motivated participants. Participants should work in groups of three; be sure to encourage them to divide up the work of cutting and taping. The following image gives a sense for the comb-disks (see participant instructions for definition) and representative spaces that the participants will be creating in this activity. (A) is a Euclidean comb-disk, which is representative of a flat plane (F). (B) is a spherical comb-disk, representative of spherical geometry, as shown in (E). (C) is a hyperbolic comb-disk, representative of the hyperbolic plane (D). Note that all the picture comb-disks have a radius of 1 unit.



Facilitator's note: Please photocopy the equilateral triangular graph paper so that there is enough for five sheets per table.

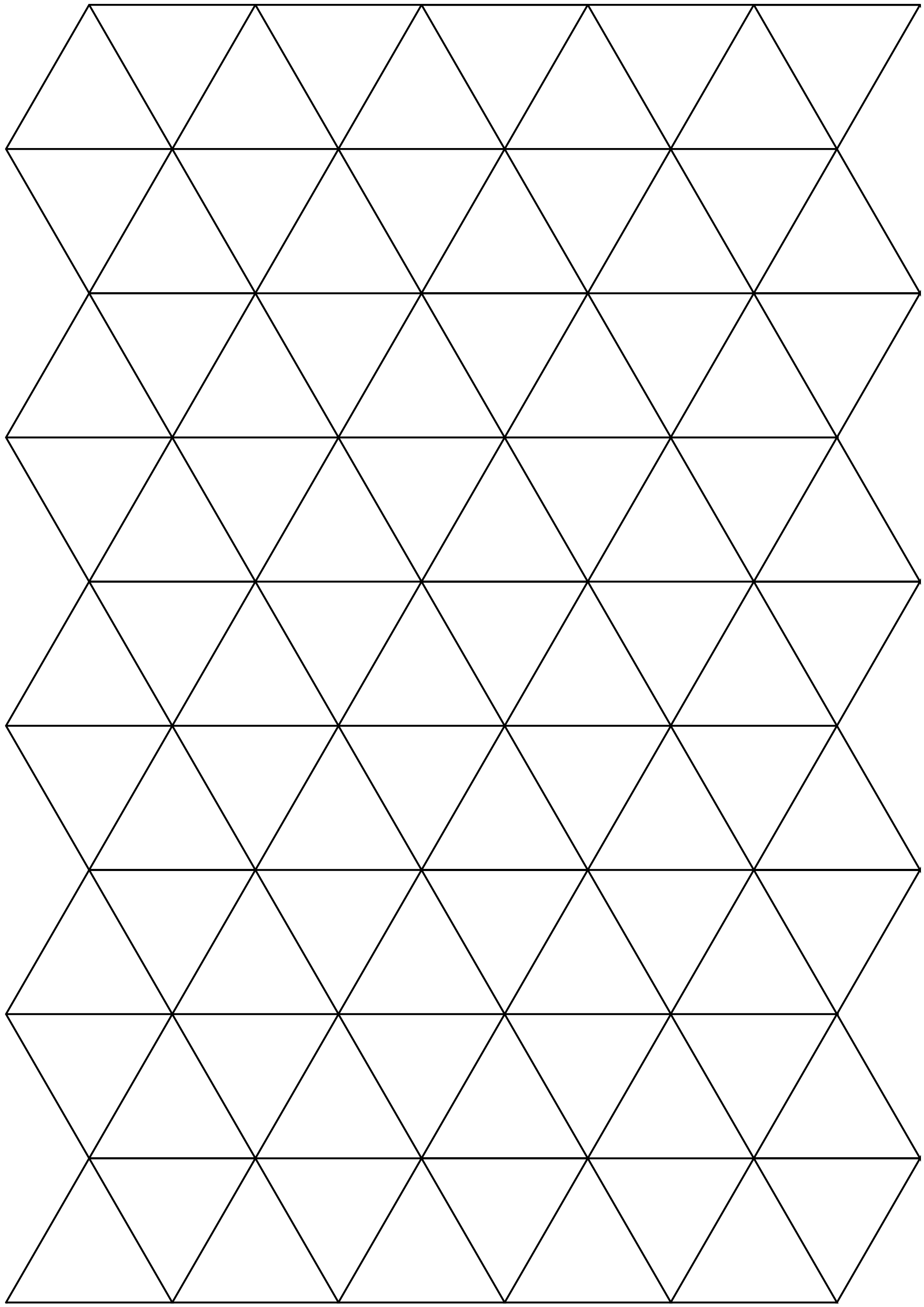
ACTIVITY 4

Equilateral Triangular Graph Paper Worksheet



ACTIVITY 4

Equilateral Triangular Graph Paper Worksheet Continued



ACTIVITY 4

In this activity you will build models of non-Euclidean geometries using familiar equilateral triangles.

Using a sheet of the triangular graph paper, draw an approximate Euclidean disk in this tiling, which we will call a "radius-*n* Euclidean comb-disk." To form it, let the radius-0 Euclidean comb-disk be a single vertex in the tiling. Now we can expand a radius-(n–1) Euclidean comb-disk to a radius-*n* Euclidean comb-disk by attaching all the triangles in our lattice that have a vertex in the radius-(n–1) Euclidean comb-disk.

1. On a sheet of triangular graph paper, draw Euclidean comb-disks of radius 0, 1, 2, 3, and 4 centered at a fixed vertex.

Answer: It will look like four concentric hexagons.

2. The circumference of a Euclidean comb-disk is the number of edges on the disk boundary. Graph the circumference as a function of the radius up to radius 4.

Answer: (0,0), (1,6), (2,12), (3,18), (4,24)—this defines a line with a slope of 6

Before the next step, remind participants that the circumference of a unit circle is 2π and the circumference of a circle of radius n is $2\pi n$.

3. Find a formula for the circumference of a Euclidean comb-disk as a function of its radius.

Answer: $C(n) = 6n$; One proof identifies the Euclidean comb-disk with a hexagon with side length n; hence, the circumference is $6n$.

Now we are going generalize our Euclidean comb-disk. Let the spherical comb-disk be built from the same equilateral triangles as the Euclidean comb-disk, by taping the triangles together such that there are five triangles at each vertex (rather than six, as in the Euclidean case). Cut out the triangles and start taping them together!

4. Argue that intrinsically this space is flat everywhere, except at the vertices.

Answer: Answers will vary.

5. What is the circumference of a small unit circle at a vertex?

Answer: $5\square\ (\frac{\square}{3})$

6. Does this correspond to positive or negative curvature?

Answer: The curvature is (2π – circumference) = $\frac{\square}{3}$, which is positive.

ACTIVITY 4

7. Form the comb-sphere by taping until you can tape no more. Can you describe the resulting shape?
Answer: An icosahedron or some equivalent description.

8. How much total curvature did you find?
Answer: There are 12 vertices, so $12(\frac{□}{3}) = 4$

9. The Gauss-Bonnet theorem (widely considered the most beautiful theorem in mathematics) tells us that the total curvature of a closed surface is 2□ π□ (Euler Characteristic). Is this theorem consistent with what we found with the comb-sphere?
Answer: Yes, the Euler characteristic of the sphere is 2, and $4\pi = (2\pi)$□ 2

10. Graph the circumference of the spherical comb-disk as a function of the radius up to radius 3.
Answer: (0,0), (1,5), (2,5), (3,0)—this requires some careful building and 121 triangles.

Let the hyperbolic comb-disk be built from the same equilateral triangles as the Euclidean and spherical comb-disks, but this time tape triangles together so that there are seven triangles at each vertex. Cut out the triangles and start taping them together!

11. Notice that this is also flat everywhere except at the vertices. What is the circumference of a small unit circle at a vertex?
Answer: 7□ $(\frac{□}{3})$

12. Does this correspond to positive or negative curvature?
Answer: The curvature is (2π – circumference) = $-\frac{□}{3}$, which is negative.

13. Graph the circumference of the hyperbolic comb-disk as a function of the radius up to radius 3.
Answer: (0,0), (1,7), (2,21), (3,56)—this requires some reasoning or some very careful building and 121 total triangles! (which is why we need to have a lot of triangles available)

ACTIVITY 4

14. (Challenging) We are building the hyperbolic plane; hence, as we saw in the video, this comb-disk of radius *r* should have loads of room. Show that the circumference of the hyperbolic comb-disk satisfies $C(n)>2^n$.

Answer: An *n*-vertex on the boundary of a comb-disk, which is the vertex of exactly *n* of the disk's triangles. We observe that each edge of the radius-(n–1) hyperbolic comb-disk contributes to the radius-*n* hyperbolic comb-disk one 3-vertex, while each 2-vertex contributes two 2-vertices, and each 3-vertex contributes one 2-vertex. Now, because there are as many edges as vertices on the boundary, we at least double the number of edges each time we increase the radius by one, as needed. From this proof, one can, in fact, give an exact number: $C(n) = 7$□ (Fib($2n$) + Fib($2n$-1)), where Fib(n) is the n^{th} Fibonacci number. So, the actual growth rate is proportional to the golden ratio squared, and □^{2} is approximately 2.6180.

15. Using the estimate from the previous problem, at least how much larger is the circumference of a hyperbolic comb-disk of radius 100 than a Euclidean comb-disk of radius 100?

Answer: More than 10^{27} times as large! To give a point of reference, there are fewer than 10^{24} grains of sand on Earth! The hyperbolic disk is incredibly huge!

16. (More Challenging) The usual hyperbolic plane cannot be embedded in the Euclidean-like space we live in (locally!). Show that for some *n* you cannot build a hyperbolic comb-disk of radius *n*.

17. Here is a so-called "open problem," one that as of this printing has yet to be solved: Find the largest-radius hyperbolic comb-disk that CAN be embedded in Euclidean space. (If you tried it, could you build the radius-2 hyperbolic comb-disk? How about the one of radius 3?)

CONCLUSION

(30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Facilitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 8, Geometries Beyond Euclid could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: Game Theory. Workshop participants are expected to read the accompanying text for Unit 9, Game Theory before the next session.

NOTES