MATHematics illuminated

PARTICIPANT GUIDE

UNIT 6

UNIT 06

THE BEAUTY OF SYMMETRY

PARTICIPANT GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

ACTIVITY 1

This activity explores in more depth some of the ideas from Unit 6 having to do with sets, groups, integers, and doing math with things other than numbers.

A

1. Under which operations is the set of positive integers closed? How about the positive odd integers? The positive even integers? Give justification for your conclusions.

B

Let S = {a, b, c, d}.
Let x = an arbitrary member of S

* symbolizes an operation that follows the following rules:

a * x = a
x * a = a
b * x = x
x * b = x
b * b = b
c * d = d * c = d
c * c = d * d = c

1. Create the operation table for * on S.

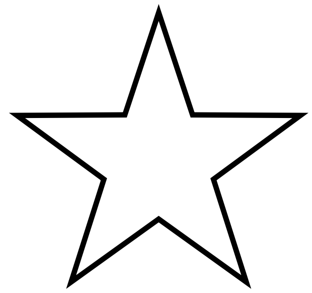
2. Is S a group under *? Explain.

C

1. Show that any combination of rotational symmetries of a regular five-pointed star (labeled I for the Identity rotation and $R_{\#\text{ of degrees}}$ for each of the four other rotations) gives one of the basic rotational symmetries.

ACTIVITY 1

2.



Create a table showing this.

D

Let's say we have a mystery rotation *x*. Use the symmetry table that you found in exercise C to solve the following equations:

Let * denote "followed by."

1. $R_{72} * x = R_{144}$

2. $R_{288} * x = R_{144}$

3. $(x * R_{144}) * R_{288} = R_{216}$

4. The system:
$(x * R_{216}) * y = R_{144}$
$(R_{72} * x) * y = I$

Hint: It may be evasier just to reason this out than to try to apply standard algebraic techniques, because we haven't defined the inverse operation of *.

ACTIVITY 2

In this activity you will explore the symmetries of various regular polygons. The symmetries of regular polygons form what are collectively called the "dihedral" groups. "Dihedral" literally means "two-sided" in the sense that a coin has two sides. In order to explore and catalog the symmetries of these shapes, you will need to be aware of both sides.

To aid in your exploration, there are marked polygons for you to cut out, assemble, and use. Each polygon has a front and a back; you should cut out both and either glue or tape them together so that the labeled vertices match up exactly.

For each shape you should identify all of its symmetries. In order to have consistent notation, you can call the rotational symmetries R_0, R_1, ...R_n and the reflection symmetries S_0, S_1...S_n. For convenience, the axes of reflection have been labeled on each shape. In order to stay consistent, all rotations should be clockwise, and let R_0 represent the identity, (a rotation of 0 degrees).

For each shape you should fill in a "Cayley" table—that is, a table that shows how the different symmetries combine to form other symmetries, sort of like a multiplication table. The table structures are already constructed for you; you just need to fill them in. One thing to keep in mind is that if you reflect a shape, then the direction of rotation is reversed (imagine watching the reflected image of a clock).

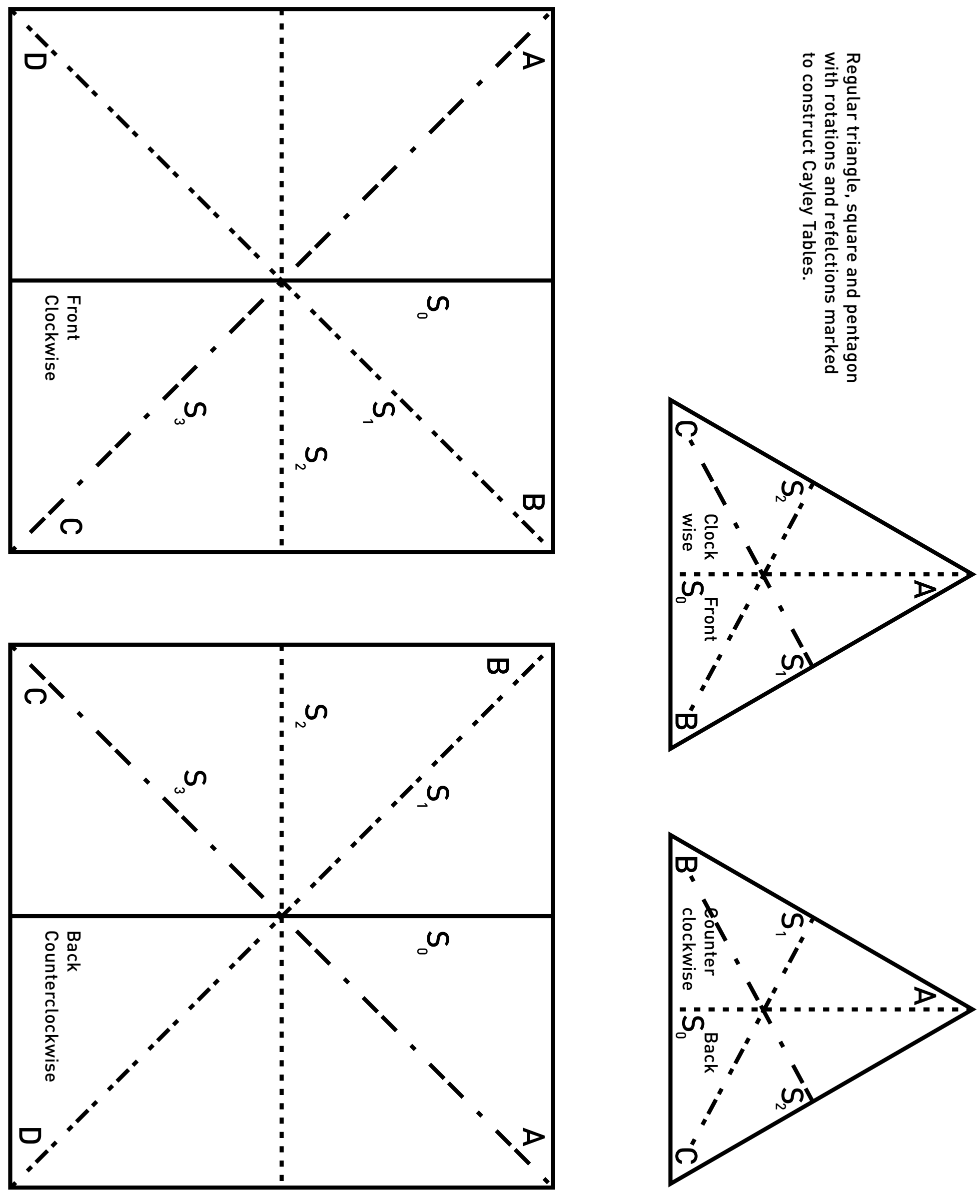
After you have completed the tables, there are some questions that will lead you to understand the group structure of an arbitrary regular polygon...an "n-gon."

Please work as a group and feel free to divide the labor as you see fit. Enjoy!

ACTIVITY 2

EQUILATERAL TRIANGLE AND SQUARE

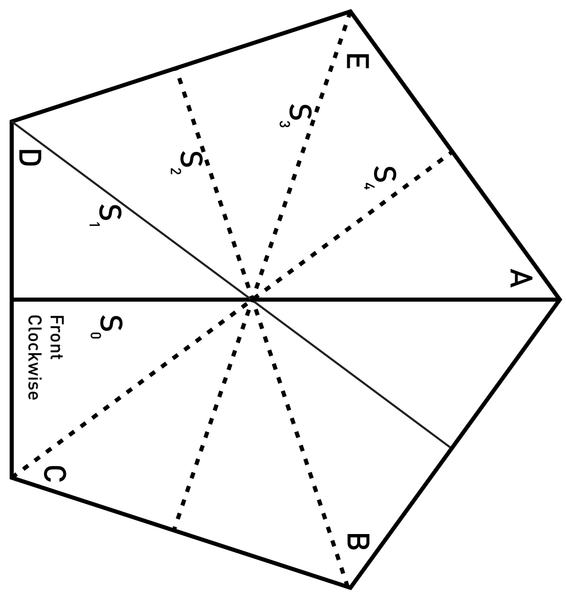
Worksheet 1 templates:

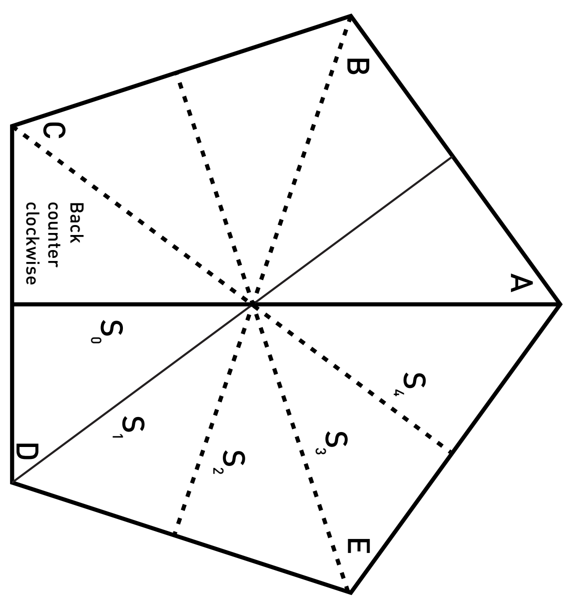


Regular triangle, square and pentagon with rotations and refelctions marked to construct Cayley Tables.

ACTIVITY 2 Worksheet 2 templates:

REGULAR PENTAGON

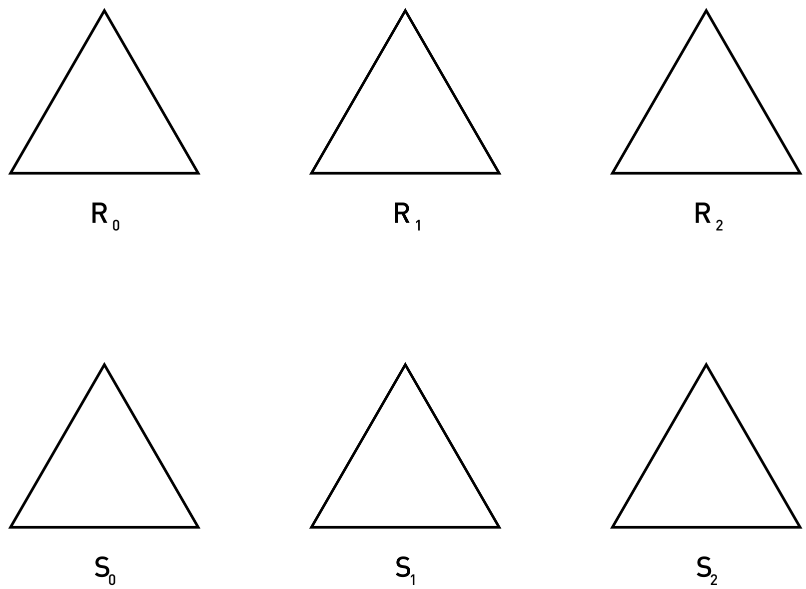




ACTIVITY 2

Worksheet 3 Equilateral triangle:

SYMMETRY TABLE



CAYLEY TABLE FOR EQUILATERAL TRIANGLE

*	R_0	R_1	R_2	S_0	S_1	S_2
R_0						
R_1						
R_2						
S_0						
S_1						
S_2						

ACTIVITY 2

Worksheet 4 Square:

SYMMETRY TABLE

R_0 R_1 R_2 R_3

S_0 S_1 S_2 S_3

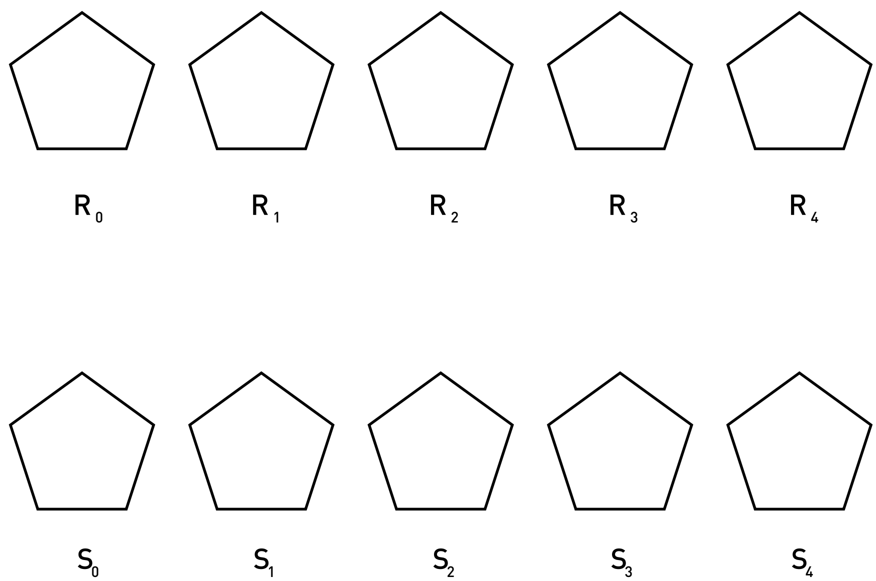
CAYLEY TABLE FOR EQUILATERAL TRIANGLE

*	R_0	R_1	R_2	R_3	S_0	S_1	S_2	S_3
R_0								
R_1								
R_2								
R_3								
S_0								
S_1								
S_2								
S_3								

ACTIVITY 2

Worksheet 5 Regular Pentagon:

SYMMETRY TABLE



CAYLEY TABLE FOR REGULAR TRIANGLE

	R_0	R_1	R_2	R_3	R_4	S_0	S_1	S_2	S_3	S_4
R_0										
R_1										
R_2										
R_3										
R_4										
S_0										
S_1										
S_2										
S_3										
S_4										

ACTIVITY 2

When you have completed the three tables, try to establish a set of rules that would allow you to construct a table for an n-gon:

1. For each table, what is $R_1 * R_2$?

2. For the square and pentagon, what is $R_1 * R_3$?

3. For the pentagon, what is $R_1 * R_4$?

4. What is the relationship between subscripts for combinations of rotations?

Hint: Think modular arithmetic.

5. Find similar relationships for any reflection followed by any other reflection, reflections followed by rotations, and rotations followed by reflections.
(Again, think modular arithmetic.)

ACTIVITY 3

MATERIALS

- A couple of old pairs of pants
- A few shoeboxes (these are not essential, but they may prove helpful for people who have trouble visualizing the manipulation of 3-D objects in their heads)

The mathematical study of symmetry leads naturally to the identification and analysis of groups, which are sets of objects that obey the four specific rules outlined in the text. Group theory provides a way to find abstract general structure in a set of objects, whether they be motions that leave a geometric object invariant or the rules that dictate inter-clan marriage arrangements. In the following activity, you will get a sense of the range of situations that can be analyzed using group theory.

A

The first group that you will explore is based on what can be done with a pair of pants (while you're not wearing them, of course). The elements of this group will be certain manipulations of the pants, as follows:

Motion X = Turn the pants around from back to front.
Motion Y = Turn the pants inside out.
Motion Z = Turn the pants inside out and back to front.
Motion I = Identity—leave the pants alone.

Since every group consists of both a set of elements and an operation, we need to define an operation for this group. The one we used in the previous activity "followed by" will work just fine. So "X followed by Y" means "turn the pants around from back to front and then turn them inside out." Notice that this leaves the pants as if you had only done Z to them. Therefore, we can say that X followed by Y equals Z.

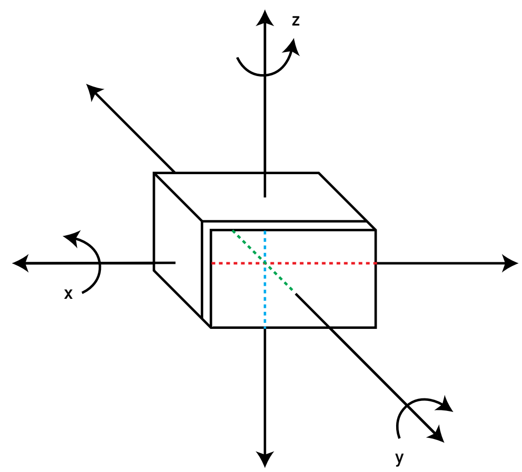
1. Construct a Cayley table for the defined motions and operations.

2. Is this a group? Justify your answer.

ACTIVITY 3

B

The second group to look at involves a certain set of motions with a shoebox. The motions to look at are:



L = a 180-degree rotation about the x-axis
M = a 180-degree rotation about the y-axis
N = a 180-degree rotation about the z-axis
I = the identity—no rotations

Explore the behavior of these elements under the operation *, denoting "followed by." For example, L * M leaves the box in the same position it would have been in after the motion N. L * M = N.

1. Complete the operation table for this set and comment on whether or not it is a group under the operation "followed by."

C

Finally, let's look at group structure in the field of anthropology. The Kariera are a tribe of Australian Aboriginal people who have a complex set of rules that determine who can marry whom. The tribe consists of four clans: Banaka, Karimera, Burung, and Palyeri. The marriage rules are as follows:

A Banaka can only marry a Burung.
A Karimera can only marry a Palyeri.

ACTIVITY 3

For simplicity's sake, let's represent each clan with a letter:

A = Banaka
B = Karimera
C = Burung
D = Palyeri

1. Use the marriage rules to create a "family correspondence." Imagine an ordered set of four people, one from each clan, {A,B,C,D}; what would the set that comprises their spouses, in order, be?

2. Let f be the transformation of {A,B,C,D} to {C,D,A,B}. This means that if f acts on a set of four members, it replaces A with C, B with D, C with A, and D with B. What happens if you apply f to {A,B,C,D} twice in a row?

3. Now consider the Kariera rules for children.

The children of a male Banaka and a female Burung are Palyeri.
The children of a male Burung and a female Banaka are Karimera.
The children of a male Karimera and a female Palyeri are Burung.
The children of a male Palyeri and a female Karimera are Banaka.

Note that the clan of the child is determined by either the clan of the father or the clan of the mother, but a child is never of the same clan as the father or mother. Imagine an ordered set of the clans of the mothers, {A,B,C,D}. What will the ordered subset that corresponds to the clans of the children of the mothers look like?

4. Let m be the transformation that, given the subset of mothers' clans, {A,B,C,D}, outputs the subset of children's clans you found in the previous question. Describe what m does to {A,B,C,D}.

5. What happens if you apply m to {A,B,C,D} twice in a row?

6. Let p be the transformation that, given the set of fathers' clans, {A,B,C,D}, outputs the set of the clans of their children. Describe what p does to {A,B,C,D} and give the resulting set of children's clans.

7. Start with {A,B,C,D}. Apply f;; then apply m to the result. What is the final result?

ACTIVITY 3

8. Could the result of the sequence of two transformations that you did in the previous question be obtained using only one transformation? Which one?

9. Let * denote the operation "followed by," and let "I" be the identity transformation. Complete the following table:

Followed by	I	f	p	m
I				
f		I		p
p				
m				I

10. Is the set {I,f,m,p} a group under *? Explain.

11. Look at the tables you have obtained for all three groups. If you let X = L = f, Y = M = p, and Z = N = m, what do you notice? What is going on here?

ACTIVITY 4

In the previous activity, you looked at transformations that take an ordered set as input and output a permutation of that set. In the text, we saw how symmetry groups correspond to permutation groups, the symmetries of an equilateral triangle matching up with the permutations of a set of three objects, for example. This final activity deals with permutations without necessarily addressing an underlying group structure. It is actually more of an exercise in probability, but it is hopefully a fun one and one that may be of interest to students.

"How many people should a person date before deciding on the one?"

The problem of finding the right spouse or life partner is one that most people at least consider at some point in their lives. With the exceptions of arranged marriages and "Love at First Sight" scenarios, people tend to meet more than one person who is a suitable spouse or life partner, some more suitable than others. The question is: after meeting how many possible partners should you choose somebody?

First, a few simplifying assumptions:

- Assume that you will meet four potential partners in your life.
- Assume that you could rank these potential partners 1 through 4, with 4 being the most suitable and 1 being the least suitable.
- Assume that you can see only one potential partner at a time, and if you reject a person, that person can never be your partner.

1. How many possible "partner-meeting" orders are there? List them.

2. What is the probability of meeting the most suitable partner on the first try in this scenario? Explain.

3. What is the probability of meeting the best partner on the last try?

Let's say that, as a general strategy, you meet and reject a defined number of people and then take the first person after that who is better than the ones previously rejected. For example, suppose that you decide to reject the first two possible partners and choose the first one after that who is better than either of those first two. The ordering 3214 then would work out this way: you reject persons 3 and 2 automatically and then meet and reject person 1 because this person is not better than 3 or 2. When you meet person 4, you stop, because 4

ACTIVITY 4

is better than everyone you have met so far (not to mention the fact that 4 is the last of the possibilities).

4. What are the chances that this strategy (automatically rejecting the first two) will result in you ending up with person 4?

Hint 1: Start by looking at all the orderings in which person 4 is not one of the first two people.

Hint 2: How many of these orderings will result in picking person 4 according to the rule "reject the first two and pick the first one after that who is better than either of those first two?"

5. Use a similar process to determine the probability of ending up with person 4 if you follow the rule: "reject the first person and then choose the next person after that who is better than the first."

6. Of the strategies discussed, which has the highest chance of success (defined as ending up with person 4)?

7. What are the limitations of this model?

CONCLUSION

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 6, The Beauty of Symmetry could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no “frontloading?”

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: Making Sense of Randomness. Workshop participants are expected to read the accompanying text for Unit 7, Making Sense of Randomness before the next session.

NOTES