MATHematics illuminated

PROGRAM: 5

Other Dimensions

Producer: Stewart Boyles
Host: Dan Rockmore

Produced by Oregon Public Broadcasting for Annenberg Media

TIME CODE	Audio
00:15	OPENING CREDITS
00;40	*SOUND ON TAPE:* *That's madness. Maybe so, but it's not half as mad as the idea that brought us to this point.*
00;45	HOST: Is there such a thing as a *higher dimension,* a parallel universe where otherworldly things can happen?
00;55	HOST: Over the years, artists, writers and filmmakers have tried to answer that question, creating some dazzling works of science fiction in the process. But are the higher dimensions we see in sci-fi really fiction?
01;09	**HOST (V.O.):** **Meet Maggie. Scientist by day, and sci-fi fanatic by night. Maggie's watching a 1950's film about scientists who invent a device that catapults them into the fourth dimension, where they can walk through walls and read minds.**
01;30	**HOST (V.O.):** **Like the rest of us, Maggie lives in a three-dimensional world. But, she also appears to have an interest in *alternate* dimensions.**
01;37	**HOST (V.O.):** **In 1940 master sci-fi author Robert Heinlein wrote the short story "And He Built a Crooked House." In the story our hero builds his home from an unfolded 4 dimensional *hypercube*....**
01;50	**HOST (V.O.):** **When an earthquake strikes, the house folds up, creating the world's first four-dimensional house. Where occupants can look down a hallway and see their own backs**
02;11	HOST: Dimension is one of the most important and intriguing ideas in mathematics. We move in 3 dimensions of space, we live in 4 dimensions of space and time, but mathematicians and increasingly, scientists of all flavors, from statisticians to biologists are finding that they need to understand and work in worlds of hundreds and even thousands of dimensions! Worlds that reach well beyond human sensory experience. When we start asking questions such as "when did our universe begin, and how will it end?" Many scientists believe the answers will involve 10, 11 12, or even more dimensions. But, what do we really mean by the word dimension? Generally speaking, the dimension of space or a problem is simply the number of numbers that we need to describe a situation. These numbers are called coordinates.
03;01	HOST: The idea of "coordinates" was developed in 1637 by French mathematician and philosopher Rene Cartesius Descartes...
03;10	HOST:

	Descartes, who never got up before 11 am, as the legend goes, was lying in bed watching a fly crawl across the ceiling.
03;18	HOST: ...and it hit him. The basis for what we now call the Cartesian Coordinate System.
03;24	HOST: What Descartes very well could have realized by watching that fly wander across the ceiling, is that the fly's position on the ceiling could be described by exactly two numbers, what we might call the x and y coordinates
03;39	HOST: Now, I have a feeling Descartes would have loved one of my favorite toys, the Etch-A-Sketch. It's all about coordinates.
03;45	HOST: With one of these knobs I can control lines that go back and forth — the horizontal direction. With the other I control the up and down — the vertical dimension. We can make some pretty pictures with this toy, but our freedom of movement is limited to just these two independent dimensions or parameters we might call them.
04;06	HOST: Now of course, we don't move in a world of two dimensions like the etch-a-sketch. We are able to move not just side to side and back and forth, but up and down as well, so that we need 3 numbers to describe a point in our physical world. So, suppose we had a 3 dimensional Etch-A-Sketch. Well, let's not suppose. I actually have one! Come on let's see it!
04;32	HOST: Here's my 3-dimentional etch-a- sketch. Now, with three knobs, I can draw vertically, horizontally, and with depth. This is so cool! I now have three independent parameters so that I have freedom of motion in three dimensions, just like in our everyday world. But in the world of mathematics, we deal with dimension in some rather unique ways.
04;57	HOST: In fact, when mathematicians talk about dimension, we're talking more than spatial dimension. We're talking about *any* set of parameters that help define everything from the markets, to the health benefits of a particular muffin, to the latest model car or even, the perfect date...
05;19	**HOST (V.O.):** **Our friend Maggie is using a computer dating web site, circa 2007.**
05;24	**HOST (V.O.):** **It starts by asking a series of 250 questions, grouped into 30 categories.**
05;30	**HOST (V.O.):** **By the time Maggie's done, she will have established values for 30 different parameters to define herself, as well as her dream date:**
05;39	**HOST (V.O.):** **With these coordinates, you could now say we're now in a 30 dimensional world. And in this world, each point represents a human being.**

05;46	**HOST (V.O.):** **And if Maggie can find someone who's coordinates are near hers, she's got a date.**
05;56	HOST: When we're talking about sets of parameters, it seems we can work in as many dimensions as we want. But what's so special about three dimensions? Because we live in a 3D world, our brains think in 3D, but can we actually see things in more than three dimensions? What would a four-dimensional object look like?
06;20	Dan Rockmore: Well, three-dimensional Etch-a-Sketches but multidimensional personalities. Can we see those kinds of things? We're here with Greg Leibon, research mathematician at Dartmouth College, and Greg, can I see four dimensions without hurting myself? Leibon: Well, you're going to have to go through a little bit of pain, I think, to see into higher dimensions, but the pain comes in the form of analogies and metaphors. So we would probably start understanding four dimensions by thinking a little bit about what we know in lower dimensions. For example, we could start with the cube. Rockmore: Okay, great, great. We've got our cube. Leibon: We've got our cube here. And there's a few things worth observing first. That looks like a three-dimensional cube to us, but of course this is on a two-dimensional screen. This is some sort of projection. Rockmore: That's true, that's true. It's a total cheat.
07;02	Leibon: But our brains, fortunately in this case, are very hardwired to interpret this as a three-dimensional object. Four dimensions is much trickier because, for example, here is a picture of a four-dimensional cube that we'll just look at for a second just to get a sense. It looks very cluttered. It's the same sort of projection where now we've gone from a cube in four dimensions, we put it on our screen down here in two dimensions, and it looks very cluttered. What it would be like as a two-dimensional being to experience a three-dimensional object, because what we are is essentially in the way we think spatially three-dimensional beings and we're contemplating a fourth dimensional object.
07;38	Leibon: Now this is not a new idea. I mean, here's *Flatland*. This is a very famous example of this turned into a beautiful story. Rockmore:

	Exactly.
07;46	Rockmore: So this is Edwin Abbott's *Flatland*, a book first printed in 1884, and it's the story of beings that live in two dimensions, on a plane, if you like, with no thickness, and how they might experience three-dimensional life, and which is analogous to our problem as beings living in a three-dimensional world trying to experience a four-dimensional life.
08;08	Rockmore: So the two-dimensional being is living on this sort of infinitesimally thin surface, right? Leibon: That's right. Rockmore: So they can move in two dimensions.
08;18	Leibon: And somebody enters their world from above, a sphere we can imagine. A sphere is this very symmetric object, and we're going to imagine it in the following sense. This sphere which actually lives in three dimensions is going to dip into this two-dimensional being's world, and the two-dimensional being is going to experience an aspect of this sphere, what we think of it mathematically as slices of the sphere, and they're going to look at the geometry of the slices over time as the sphere moves through their world and they're going to use that as a summary statement as to what three dimensions look like.
08;47	Leibon: And as the sphere dips through the space, of course, this point instantaneously blows up into a little circle.
08;51	Leibon: As the sphere dips into the plane, it grows, it gets to some critically large length, which is of course the equator circle of our sphere, and we see it comes back and it shrinks back down to a point.
09;04	Leibon: And so what has the person experienced, who lives is this plane is a little movie of some circle getting bigger and collapsing back down to a point. And this is their way that they're going to initially think about the two-dimensional sphere that lives in 3-D that we're used to. Is they're going to think about it as a bunch of slices at different times, and they're all put together by the continuity of time into an object. And we see that object here because time we can think of as our third axis, our third dimension here. As it dips through there, that dimension changes.
09;33	Rockmore: So there's a sense in which we see the movie all at once, but they have no ability except to see one frame. Leibon:

	That is exactly the analogy that we want to have in mind. Because now we're going to imagine looking out into our room here. And now we're going to have something that starts of as a point. It expands into a sphere, it reaches some critical size, it shrinks back down to a point. And what have we just seen? Well, to us we just saw a movie of a little explosion and contraction of some kind and we're not sure what we've seen.
10;01	Leibon: Now, it could well have been that the three-dimensional sphere in four dimensions has just dipped into our world to give us a visit, okay? And now we understand that analogy because we can experience exactly what happened in the two-dimensional case, and that's the power, of course, the *Flatland* analogy, is that now something that we can witness in our world, we can now understand that via this analogy of the Flatlander experience.
10;25	Leibon: So what you have just experienced is an object that lives naturally in four spatial dimensions: the three-dimensional sphere in four spatial dimensions could be thought of as this little movie as you might experience it and as our Flatlander did experience it.
10;39	Leibon: The problem is we can't step back and see the movie all at once, okay, and we don't do that very naturally. But through using this analogy we can get a grip on what higher dimensions might -- well, how they really do look in our lives. Rockmore: Right, yeah, and a way to start to begin to think about them as mathematicians. Leibon: That's correct. So now maybe we should go to the more fun object, which are these cubes. Rockmore: Great. Leibon: So now let's imagine once again we're in our *Flatland* world, and now a cube is going to dip into this world.
11;05	Leibon: Right. It's nice to look at this trivial direction where it comes in and all of a sudden, instantaneously, we see a square in our two-dimensional world. It floats through and it's totally static-looking, it looks like a static square that just suddenly disappears. Rockmore: Yes.
11;17	Rockmore: So now we can get a different view, or we can let our flatlander get a

	different view of the cube by actually changing the way in which the cube comes into his world, right? Leibon: That's right. Our original way where it came down with the flat -- first gave us this boring movie, and in many ways it's sort of a boring look at the cube as a three-dimensional object. It's gleaning very little from it. But if we bring it down from another angle, the Flatlander learns a lot about the geometry of the shape, so let's come down vertex first. And as we come down kind of at a slight angle here, but we can watch it on the screen, this vertex when it comes vertex first he sees a point, just like in the spherical case, a point.
11;50	Leibon: But that point, rather than exploding then into a little circle has now become a little triangle, okay? And this triangle as we lower it down, every time one of the vertices of our cube in three dimension hits it, an edge is born and we see this birth process of these edges as our triangle. It changes into this quadrilateral, which first has this very short side, which gets longer. And then it changes into this pentagon-type shape, five-sided object, into this hexagon.
12;15	Leibon: And now of course this hexagon has gone back to a pentagon -- or a pentagon-type object, five-sided -- then to this quadrilateral that we see, finally going down to this triangle and a point. So it's a lot like the sphere case.
12;29	Rockmore: Greg, what I like so much about this discussion is that it's actually part of things that happen in the real world all the time. For example, when people are trying to image your body, they actually do it by constructing slices of you using some kind -- you know, like x-rays or whatever it is, and they rebuild your body from those slices. And the other way in which it's come up is in Cubist art, in fact.
12;50	Rockmore: And Picasso's idea when he was creating this was actually inspired by discussions of how to see four-dimensional things as three-dimensional slices, which is how we're actually going to try to see four dimensions, like the Flatlanders tried to see three. Leibon: Yeah, absolutely. So it's easy to see how art and this would interact in the sense that your understanding here is very much so a metaphor in the same way that the art might be used. But let's look at it and see what we're talking about.
13;13	Leibon: Here we're going to be playing that same exact game where we dipped our cube into the Flatlanders world, but we're going to dip a hypercube into our world and see what we see.
13;23	Leibon:

	the cube that lives in four-dimensional space, at least a projection of its
13;28	Leibon: So what are we watching? We are watching a four-dimensional object dipping into our world. Here it starts, we get our vertex. As it dips in, now we get to a simplex, a tetrahedron with four vertices as it comes in the corner. And now watch it evolve. As it hits these corners, the vertices are chopped off and we get these things that have faces that look like our original quadrilaterals -- Rockmore: Right. And the perfect analogy to what was going on for the Flatlanders.
13;54	Leibon: And now we go through the whole sequence and we see that that shape changed over and over again until we're back at our vertex. And we have a beautiful movie that captured this object in four dimensions, this rigid object, which was perfectly stationary, four-dimensional object, has now been experienced by this little film. And in fact a great deal of its geometry can be understood by dwelling on this, just like the Flatlander, if they were so inclined, could understand a lot about this cube they were looking about by understanding their movie. And these analogies and metaphors are very important for getting our initial glimpses into fourth dimension. And in fact, we're looking at probably the one that is the most common way to introduce fourth dimensions now and is, of course, time.
14;35	Rockmore: And I think Maggie's also going to use time to help us see four dimensions. So let’s go take a look. Leibon: Great.
14;41	**HOST (V.O.):** **In 1910 Scientific American held a contest, asking readers to describe the 4th dimension. The 15 winning entries went into great detail about a 4th spatial dimension.**
14;52	**HOST (V.O.):** **But not one entry mentioned the other fourth dimension — the dimension of time.**
14;58	**HOST (V.O.):** **Albert Einstein changed all that when he put forth his theory of relativity. Einstein said you can’t talk about space without dealing with time.**
15;08	**HOST (V.O.):** **Looks like Maggie has an appointment at the dating service...**
15;11	**HOST (V.O.):** **...which appears to be on the corner of “X” and “Y”...**
15;17	**HOST (V.O.):** **...she needs to go up to the** **Mezzanine level...**

15;19	**HOST (V.O.):** **...and be there by one o'clock.** **MAGGIE:** **Hi I'm here for the speed dating.** **RECEPTIONIST:** **I'm sorry it started at one o'clock.**
15;30	**HOST (V.O.):** **But since Maggie has arrived thirty minutes late, she's got the wrong coordinate in the time dimension…**
15;37	**HOST (V.O.):** **...and her on-line dream date just walked out the door with someone who showed up on-time.**
15;47	Rockmore: So the fourth dimension: time? I don't know. Maybe. But in fact, as far as a mathematician is concerned, it could sort of be any kind of statistic, right? It could be height, weight, I mean, whatever you like, any parameter, so to speak. Leibon: Yeah, you're free to have 30 of them, 30 of these parameters. You're in some big 30-dimensional space where your coordinates and your parameters -- Rockmore: Which we can't see. Not even you can see.
16;10	Leibon: Yes. And in fact, in some ways see is kind of the wrong thing, and part of that is because these things might have very different sorts of units. You know, like space, it's all in the same unit of measurement, and so you can do all kind of cohesive things: make measurements and do things that are across these dimensions in a sensible way. And there are notions of dimension that have everything to do with that: objects living in a place where there really are units that are -- Rockmore: I see. So we're going from a notion of dimension as simply number of parameters to almost a measurement.
16;38	Leibon: Maybe we should start with a simple example, okay? We're going to imagine that I have a ruler, and my ruler is a foot long, and you have a yardstick.
16;46	Leibon: Let's build a line. Rockmore: Okay. So my line is three times as long as your line.

	Leibon: That's right. So now let's build a square. Rockmore: If I use 20 rulers and you use 20 rulers, I've still got a line three times as long as your line. Leibon: That's right. So these things are going to scale perfectly. So now let's suppose that you go ahead and make a square out of your yardstick with one of your units on each side. And we have this square here. And now my question is I'm going to build the same thing, but in my units I have this foot long ruler.
17;13	Rockmore: So my square is nine times as big as your square. Leibon: That's right. Your square is nine times as big, or three squared. It was -- the line was three times as big; this is three squared. Now that notion of dimension of square that's coming in is our next notion of dimension that we're -- Rockmore: Two-dimensional square in my -- Leibon: That's right. How many more of my objects is it going to take to fill yours? It's a spatial notion. Now let's build a cube. Rockmore: Okay. Leibon: Now here's your big cube and my little cube, and this is a problem that people that make statues run into all the time. How many of mine are going to be taken to put inside of yours? Or what is it?
17;44	Rockmore: Right. And I'm betting it's going to be three cubed. Leibon: That's right. Rockmore: Because if my ruler is three times as large and I'm going in three dimensions now, so it's going to be three to the three, so 27 of your cubes are going to be in mine. Leibon:

	So we have these 27 cubes filling up that space, and this is a very important thing the way that an object's scale, how it's scaling the sides. If I make my ruler three times as big, it takes 27 times as much. Here's a fun one that brings us back to where we were before. Let's take a hypercube. You're going to build a hypercube. Rockmore: Build a hypercube out of a yardstick.
18;14	Leibon: Out of a yardstick. So you're some four-dimensional being who has this luxury. You've built the hypercube and now we're kind of used to looking at this picture. It represents something four-dimensional
18;22	Rockmore: So I'm going to guess, okay, that I'm going to need three raised to the fourth power of your hypercube to fill up my hypercubes. Leibon: That's right. Rockmore: So three to the fourth is 81. So that's -- that's my guess.
18;31	Leibon: And let's watch. 81 of my cubes fill up your hypercube. So this is -- Rockmore: It's all true. I couldn't have seen it, but it's all true. Leibon: It's all true. And this notion that we're using an analogy to understand dimensions is going to be very powerful for our next understanding because we've just gone through some nice friendly dimensional counts. Let's build something a little bit more interesting. Let's build a snowflake, something called a Koch snowflake.
18;51	Leibon: So what I'm going to first have you do is take your yardstick and build an equilateral triangle, all the sides are the same. Rockmore: That I can do. Leibon: That we can do. And now I'm going to have you take each side of that and split it into three equal sized pieces, which now happen to be the size of my ruler because there's three feet in your yard.
19;06	Leibon: You can take that middle piece, and you're going to remove it. And you're going to replace it with the edges of the equilateral triangle that would have that as a base, so now we see this little star thing.

19;15	Rockmore: So I'm adding more points. Leibon: So it's more complicated. Nowhere near what you might think of as a snowflake, but now we're going to do something in math we call iterate. It means we're going to repeat this process. So you're going to take each one of your edges, and you're going to do that again.
19;27	Leibon: You're going to remove the middle piece and now you're going to do something that mathematicians like to do; you're just going to do it forever. Rockmore: Right. Okay. That is the mathematician's credo: take it to infinity. Leibon: That's right. We love limits.
19;37	Leibon: And this snowflake here, now we're going to be playing a similar game, is ask: I'm going to build the same snowflake but using my ruler.
19;44	Rockmore: They seem to only differ in scale. Leibon: They seem to. Now, let's see, in before, your line took three of my lines, your square took nine of my squares. Let's see how many of my snowflakes it takes to fill in top of yours. Now, fortunately, the side of your snowflake, the iterative construction, looked exactly like mine because you were one-third the length after one step.
20;06	Leibon: So I can unfold my snowflake and put it on top of yours, and I don't cover much of it. I cover, well, about a fourth of it. And if I built three more, and I put them on -- Rockmore: Right. Folded those, yep. Leibon: I see I've filled your snowflake with four of mine. So now we're in a funny thing: the line took three of mine. Rockmore: That's right. Leibon: The square took nine. And now I have four, which lies between three and nine.

20;31	Rockmore: Right, and -- and so now what we're interested in is that sort of relation, so three to nine is a relation of squared, but three to four is actually a relation of a fractional power. Leibon: That's right. Right now we want -- Rockmore: Something in between one and two. Leibon: That's right. We -- that four is equal to three raised to the dimension! Okay? And we need to find that dimension.
20;51	Rockmore: Right. Leibon: Well, that requires solving that equation. Fortunately, with mathematics, we can solve that equation and we get something like one and a quarter dimensions. You know, some fractional dimension. So this gives us a sense that dimensions don't necessarily have to be integers, and this notion of dimension is very powerful in all sorts of ways.
21;08	Rockmore: Fractional dimension or the short -- fractal. Leibon: That's right, this is called a fractal. Rockmore: This is an example of a fractal, and life is fractal, in fact.
21;16	Leibon: That's right. Rockmore: That's right. And in fact, fractals are used to create all sorts of computer graphics, not just this snowflake here. So all sorts of things that look lifelike in some sense: mountain ranges, seacoasts, all those things look kind of fractal.
21;31	Rockmore: So, Greg, thanks a lot for the explanation, and we're actually going to see it in action right now. Leibon: It's my pleasure. Great. My pleasure to be here. Thanks for inviting me, Dan. Rockmore:

	Great.
21;39	Karl Richter: My name is Karl Richter. I work at Laika/house. I'm a technical director.
21;48	Richter: Laika is a production company in Portland. I work in the house division, which does television commercials.
21;59	Richter: We do CG commercials, which is computer animation, as well as stop-motion animation and hand-drawn cell animation as well.
22;09	Richter: My particular role is to really bring other people's ideas to life. And so, you know, a director will come to me and explain their vision for what's going to happen, and then my job is to really realize that and make it exactly how he wants it to be.
22;27	Richter: Being as we use computers to generate all of our computer graphic elements. All of these engines are driven by math.
22;37	Richter: One of the simplest examples of how we use fractals is called fractal noise. Fractal noise is just kind of an organic pattern that we can introduce to an element to make it seem more natural.
22;55	Richter: You know, things like this might be like mountains. If you look at a mountain range, you can see like big mountains, but then when you look closer there's smaller mountains. Another thing that is like this are clouds. They all have like very big details and very small details. Fractals are perfect for describing something like this.
23;14	Richter: I already have like a small cloud bank here that I've set up. And basically what this is doing is this is running a -- a fractal algorithm in a 3-D space, and we're just kind of taking and looking at a little cross-section of this 3-D space.
23;32	Richter: If we go into the texture part. This section right here defines the actual equation that's being used to generate these clouds, and this is a fractal equation that this three-dimensional space is kind of taking a slice out of.
23;51	Richter: To illustrate this if the texture isn't being applied at all it looks basically like this flat uniformed fog bank, and the more I turn up the area where these basically the negative space were these, this a algorithm is describing, the more these clouds are revealed.
24;19	Richter: This is a short that the studio did called "A Dream for Hokusai,"
24;24	Richter: And it's primarily stop-motion. After we were done doing all the stop-motion animation, they handed it over to me to add some computer graphic effects.

24;33	Richter: So the first thing I added was this mist that comes in over this water, and this water is actually just a piece of mylar that's been stretched out across this table here. And this mist was added using the same kind of fractal noise that we were using to do the clouds. I just added kind of like some subtle upward velocity to it and some better lighting as well.
24;58	Richter: As we zoom into the forest and these samurais that are fighting, you know, there's these leaves that were added in
25;05	Richter: The motion of these leaves fluttering as he's like swiping his sword is all defined by using a turbulent noise field, which is basically using fractal noise to interfere with the vectors of each one of these leaves to create -- give it this natural kind of fluttering motion.
25;28	Richter: I think the whole idea of this animation -- was to bring these paintings to life and to kind of imagine what happened in between, like, this early morning and then this winter night shot.
25;42	Richter: I really like the whole idea that we're kind of creating stuff from nothing in a way. For me it's really exciting, to bring to life elements that really don't even exist.
25;54	Richter: It's awesome that we're able to bend math to such artistic ends.
25;00	**HOST (V.O.):** **In today's high tech world of special effects, video games, and virtual reality, the mathematics of "dimension" has, well, taken on new dimensions -- as fractals have become an important method of generating realism in complex natural objects for animators. Still, understanding dimension is truly a powerful means of experiencing our real world in ways that help us solve great mysteries like the nature of time and space. And simple problems, like finding the right date.**
26;30	Host: So what's the answer? Is this real or just science fiction? I can't tell you for sure, but one thing I can say is that in the world of mathematics, higher dimensions are not fiction at all. They are reality.
26;50	CLOSING CREDITS