MATHematics illuminated

PROGRAM: 4

Topology's Twists and Turns

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Host: Dan Rockmore

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Time Code	Audio
00:15	OPENING CREDITS
00;40	HOST: Can you imagine the shape of the universe? What if we could see it from above? Or from below?
00:53	HOST: Or maybe from over there?
00:57	HOST: Or maybe from inside?
01:01	HOST: Of course, we are already inside the universe. And this is just what we imagine part of its shape to be. And mathematicians are imagining and discovering the shape of everything from the cosmos to the micro cosmos, including the smallest strands of DNA.
01:18	HOST: So welcome to the world of Topology.
01:23	HOST We can begin to understand this most amazing branch of mathematics with a simple game of Asteroids.
01:31	**HOST (V.O.):** **Consider the universe of a video game. On the surface, it looks like a two-dimensional square section of the standard Euclidean plane that we that learned about in high school. But something interesting happens when our little ship jets away to avoid a collision.**
01:47	**HOST (V.O.):** **What just happened? Did the ship leap into hyperspace? Did it fall off the edge of its universe? Are the programmers just messing with our perceptions, trying to make the game more difficult? Or is there something else going on?**
01:59	**HOST (V.O.):** **Well, something else is going on. In reality, the ship is simply traveling around its universe...**
02:05	**HOST (V.O.):** **... going on a perfectly continuous round trip, blasting asteroids to do it safely.**
02:12	HOST: For us, viewing the game on a video screen, the asteroid's spaceship appears to live on the surface of a two-dimensional plane. But actually, it could be "living" in other places, like the surface of a cylinder, or even more exotic surfaces!
02:27	HOST: That's where Topology comes in: a branch of mathematics concerned with the study of spatial relationships that don't depend on measurement, and is more concerned with concepts like 'between' or 'inside,' and how things are connected.

02:41	HOST: A topologist doesn't care that we bent the Asteroid universe to better understand it. In fact, to a topologist nothing happened. The game's universe is still the same; it's just viewed from a different perspective. This "looking at the world from a different perspective" is at the core of Topology, and today helps mathematicians get to the essence of many unusual puzzles. And it all started with seven bridges.
03:11	HOST: In the early 18th century, one of the world's greatest mathematicians pondered a popular puzzle of his day. It was a puzzle about the town Konigsberg, a town that was cut through by the river Pragel, and right in the middle of the river was a small island Konepaph. The island was connected by seven bridges, and the towns people liked to ponder whether or not you could have a trip that actually used each bridge exactly once.
03:35	**HOST (V.O.):** **Now this is a problem that you might try to address exhaustively, looking for all possible paths and seeing if any of them would work.**
03:42	HOST: But in fact it was a problem that you could solve mathematically.
03:45	HOST: And the mathematician who figured this out was Leonhard Euler. Who actually lived in Konigsberg. He realized that the problem wasn't one about "geography." It had nothing to do with the lengths of the bridges or their distances from one another -- but it had *everything* to do with connectivity. Which bridges are connected to which islands or riverbanks. He got to the essence of the problem by simplifying it.
04:08	**HOST (V.O.):** **Turning the bridges into lines, squashing the landmasses into points, which we'll call vertices.**
04:14	**HOST (V.O.):** **And in the process, he formed a graph, which became the topological backbone of the problem. Euler had turned his trip around town into navigating a graph of 4 vertices and seven edges – thereby transforming the problem into one of graph theory. And thereby creating the field of graph theory and topology.**
04:34	HOST: In his solution, Euler realized that -- other than the vertices where you start and end your path – every time you take an edge into a vertex, you have leave by another edge in order to avoid crossing the previous one again.
04:46	HOST : So that means: if you're never going to double back, the number of edges coming out of each interior vertex must be even.
04:54	**HOST (V.O.):** **But Euler's graph of the Bridges of Konigsberg revealed that all its vertices were connected to an odd number of edges - so there could be no route that crossed all seven bridges without crossing one twice. In other words, such a trip was impossible.**

05:09	HOST: In looking at the world from another perspective, Euler took geometry out of the problem altogether, and turned it into a simple problem of connectivity. He discovered by that ignoring the geometry he was able to get to the essence of the problem. And in fact, this is a process we are quite used to: A globe seems like a reasonable representation of the earth, even though most of us will never really see the entire sphere itself.
05:34	HOST: To us, locally, it essentially appears as something in two dimensions, flat, like a surface.
05:41	**HOST (V.O.):** **But if we pull back far enough, the small, bounded local surface ultimately becomes the Earth, which is a sphere -- one of the basic topological shapes and again an example of a 2-dimensional surface. Now because we've seen it from outer space, we know the Earth is a sphere, but...**
05:58	**HOST (V.O.):** **...what if we had pulled back and found that it's more like a donut? Now this is entirely possible because a doughnut, like a sphere, is locally two-dimensional and might appear flat. Now a mathematician might call a donut a torus and it's another of the basic topological shapes and an example of what we call more generally, manifolds**
06:18	HOST: Now you might be resisting such an "alternative view" of space and more generally, shape, but the fact is, it is only with topology that we can distinguish between the large scale structure of different shapes that on the surface, seem the same, like a piece of a torus or a sphere. And conversely sometimes we're very surprised by things that may seem very different, but are topologically the same. For instance, what if I told you that a coffee mug like this is essentially the same as a donut? Well, it's true: topologically, we can manipulate it, we can deform it so that it ends up just having a single hole surrounded by a single uninterrupted surface. Just like a donut.
07:01	HOST: However, it turns out that you can't turn a sphere into a torus without ripping its surface.
07:11	HOST: So we're only interested in the surface of this. I push down, I push down. And now the upper and lower sides are touching and the only way to get a hole is to in fact rip it.
07:17	HOST: So now we have a pair of surfaces: a sphere and a torus. These surfaces may in fact appear the same to a creature who lived on them. Each provides a different model for a two dimensional universe. So what kind of universe does our game of Asteroids live in? A universe in which little ships sail off one edge only to return on another?
07:40	**HOST (V.O.):**

	We know we're looking at a flat plane, because that's what a video game screen is. And we know we can glue the top and bottom edges together to create a cylinder for it to play on. But we can also put the game on a torus:
07:53	**HOST (V.O.):** **Again, the ship doesn't know it's on a torus. It just knows it's blasting asteroids out of its way. For us to play the game really well, we just have to re-orient our thinking a little to be part of its global topology and not be tricked by its local topology.**
08:10	**HOST (V.O.):** **But are there other universes for our game to play itself on besides a sphere and a torus? And if there are, what might those be like?**
08:19	Dan Rockmore: Well, this is what can happen if you're not an expert in topology sometimes. I'm very happy to be here with Greg Leibon, a mathematician at Dartmouth College
08:27	Leibon: Thank you, Dan. I'm happy to be here.
08:30	Rockmore: Our poor topologist, it turns out he was on a torus, but in fact he could have been on many other kinds of surfaces, right? Leibon: That's right. So there's one – when you go to classify it or think about what are the possibilities of surfaces. There are two major kinds especially, in terms of what you need to get used to to think about them.
08:44	Rockmore: So there's a first distinction there: orient able surfaces and there are non-orient able surfaces.
08:48	Leibon: That's right. And it's nice to think -- maybe first the simplest example is non-orient able.
08:54	Rockmore: The most famous example, in fact.
08:55	Leibon: This is the most famous example. This is the Mobius strip. And this is a -- well, if we imagine that this has no thickness -- you know, we have to keep those things in mind. And maybe one other thing, it has a boundary, which means it has some kind of edge to it. In fact, this is a fun thing to right off the bat is you follow this edge around. It only has one edge,
09:11	Leibon: So it's important, we're thinking intrinsically here. We're living inside the surface and we take -- and now we're going for a little walk with our clock. We are heading around, we come back, and we are a little bit surprised to find that it's no longer going clockwise; it's going counterclockwise, and this would be something that would surprise you if you were living on the surface potentially.

09:33	Rockmore: Right, and that's the non-orientability somehow, that you came back and there wasn't this consistent definition, if you like, of clockwise.
09:38	Leibon: And in fact, to a mathematician, that's how we define non-orientability, is there's such a path, which is equivalent to meaning there's a Mobius strip living inside your surface somewhere - that there's one of these paths that flips over, and if you look at your little band around you, you're a Mobius strip. And that's how we think of what a non-orientable surface is.
09:54	Rockmore: So we've got this initial division of surfaces: there is orientable and there is non-orientable.
10:00	Leibon: That's right.
10:01	Rockmore: And so now having done that division, we can now talk about what kinds of surfaces there are.
10:05	Leibon: Actually to look at another version of the mobius strip that is a little easier to do what we want to do woth it. This is called the cross-cap. And it has a nice property now, namely that this one boundary circle that is all confused and moving around in space here after this is a nice regular flat, round circle in a plane.
10:21	Leibon: And this advantage is that I can take this cap, and now put it on your head.
10:25	Rockmore: You can! It might actually catch on as a style
10:29	Leibon: (Laughs) Well I mean…. I could take for example our sphere here. I could cut out a little disc. That has another one of these circle boundaries. And I could stick on a cross cap. And in fact the resulting shape even has a name after I do that. It's called the projective plane, when I stick a cross cap.
10:44	Rockmore: Right. And that name comes from an interesting identification that this object that you've created has. As with some other object which in fact was historically important in the arts. Leibon: Yes very much so. Rockmore: Projective geometry. Giving pictures a sense of depth
10:58	Leibon: So here's another good example. If I stick two cross-caps together, or another one on this sphere, I get another famous object called the Klein bottle. And this is another way of seeing it.
11:08	Rockmore:

	Another classic mathematical shape. Leibon: Another very classic mathematical shape.
11:11	Leibon: Now, one very nice thing is once we have the cross-cap, or the projected plane, and the Klein bottle, we don't need to know a lot -- any more non-orientable surfaces by name. We now need to get to the orientable ones.
11:24	Rockmore: Right. You've actually done most of the mind-bending at this point, in terms of surfaces.
11:30	Leibon: That's right. And the hard intellectual work in this, at least for making the list of surfaces. So now let's get to the part that's not quite as mind-bending, and this will be the orientable surfaces, because each one turns out to be equivalent to something very reasonable to think about.
11:42	Leibon: So I start here with my ball, and now I'm going to put a handle on it like a purse handle, so it'll be like a little torus handle. So maybe we see it on our screen. Now we had to cut out two holes and we stick on this thing that's called a handle to a topologist. Now, we notice that you can deform it into a perfectly normal looking torus, a donut, and they're topologically the same. The reason why we want to think of it as having this handle structure is it's easy to imagine putting on more.
12:09	Rockmore: And you just keep doing it and you keep doing it.
12:10	Leibon: And so if you put N handles on there, you get an infinite but very controlled list of surfaces.
12:16	Rockmore: More and more elaborate donuts, so we go from a donut --
12:19	Leibon: To a two-hole donut to a three-hole donut. And the nice thing about the many-handled purse, it's an infinite list, but it's infinite only in the way that integers are, something we're very comfortable with. I just put one more on one more on one more on, and that is the list of all orientable surfaces.
12:32	Leibon: One of the hardest things about this is realizing how even that statement is amazing, because if you look in the world, like look at this chair. Here's a chair that you could imagine sitting down on, and it's orientable. It’s a very nice chair
12:45	Rockmore: Right. Okay, so we've got our chair here, and so now we know, in fact, that to a topologist it's something on that list.
12:51	Leibon: It's something on this list, but look at it. Who is it on that list? It's shaped in a very different way. If it's an embedded surface in space it's on this list.

12:57	Rockmore: Right, so we've got to start squashing and stretching.
13:00	Leibon: So let's do it. So we're squishing up the legs. They smooshed together. Oh, we're folding it out. Yeah, we see these holes. Oh, the top's coming down to put us in a nice position. And now it's becoming very symmetric, and we see -- and lo and behold, now it doesn't look like our handles, it looks like sort of a donut with six holes all in a row.
13:19	Rockmore: Yes, yes.
13:20	Leibon: But in any case, I think the point of this is that -- or at least the point to me is that you look around the world and you see, oh my gosh, the notion of surface is everywhere. All these objects are some kind of surface, but they're really complicated. Which ones on this list are they? Who are they?
13:35	Rockmore: So now, you had said before that I have this list, and you said I should take your word for it that they're all different, but you're not going to give me a way.
13:41	Leibon: I'm going to give you a way! Because it would be pretty - Mathematicians wouldn't like to take each other's words for it on these sorts of things. We would like a proof.
13:49	Rockmore: We're bad about that, we're bad about that.
13:50	Leibon: And in fact, we're extremely lucky here. We get to associate what's called an invariant. So what this means is it's something that doesn't change when I deform it, so something that --
13:58	Rockmore: Which is the thing I want, of course, because I'm trying to say two things -- that these two things can't be deformed, in fact, to each other.
14:04	Leibon: You hit it on the nose this time; it's a single number that you can attach to them, and in fact it's probably the single most famous of all of what we call invariants, or topological invariants. It's called the Euler characteristic, and it goes back --
14:15	Rockmore: Euler, the same guy with the bridges of Konisberg?
14:18	Leibon: Same guy with the bridges of -- very fundamental at the foundations of topology. And the Euler characteristic is a number, and we have to get to that number somehow. And the way we get to it is that we break our surface up into little pieces. Like in this case we have a torus, or a donut, and we've broken it into little quadrilaterals in this picture.
14:38	Rockmore: So it's as if it was constructed by tiles.

14:41	Leibon: Now, once you have those pieces, the count is the following, is that there are things that we are going to call faces, edges, and vertices in that construction.
14:48	Leibon: And we can count them. So in this case, we have *many*, right, if you look at it. Now, you can also count the number of edges.
14:56	Rockmore: Right, so it's essentially just the lines that I'm seeing here that demarcate where the boundaries are.
15:00	Leibon: That's right. You're just counting the lines on that object. And then the vertices are just the points on that object.
15:03	Rockmore: Just the meeting points
15:05	Rockmore: Okay. So I've got these three numbers: the number of faces, the number of vertices, and the number of edges.
15:09	Leibon: And if you take the faces minus the edges --
15:12	Rockmore: Subtract the number of edges.
15:13	Leibon: Plus the vertices --
15:14	Rockmore: Add the number of vertices.
15:15	Leibon: Then you have a number called the Euler characteristic.
15:17	Rockmore: Uh-huh.
15:18	Leibon: And it has the properties invariant, and what that means is no matter how I do that, no matter how I break my object into polygons, that number is always the same. It's always the same if they're topologically the same, and it allows you to decide the two surfaces are different. Now, it happens that it even has a nicer way to think of it, is that our purse has these handles, and you could think of in the orientable case, two minus two times the number of these handles that are stuck on the surface.
15:46	Rockmore: Those are going to be the same numbers.
15:48	Leibon: Those are going to be the same number, so that handle count is immediately related to this number. Rockmore: So I take two and I subtract twice the number of handles
15:54	Rockmore: So the torus, for example, so one handle, two minus two times one is zero.

16:01	Leibon: Zero. You're always going to get zero.
16:03	Rockmore: So the Euler characteristic of a torus is zero.
16:04	Leibon: Maybe this isn't the easiest way to see that. But it gives you power in places like our chair. I can just break the chair up into polygons, do this count, and I'll know it's the six-holed torus without going through that complicated deformation process which is a little bit harder to understand, So this is an incredibly powerful tool, and topologists are using invariance all the time now to study problems that are sort of a fundamental connection between topology and then with combinatorics and algebra.
16:29	Rockmore: And I have to say, the first time I saw that, that was really magic that you could calculate this number for the cube, you would calculate it for the icosahedron, for the tetrahedron, all these beautiful platonic solids, and this number was the same for all of them.
16:40	Leibon: Yes, in life it's extremely rare to get a perfect invariant, anything perfect. But the Euler characteristic is about as close as you get. It's one of the reasons why this is a very beautiful theory to mathematicians. It's so complete. It's really a truly complete list.
16:55	Rockmore: So now we've got this list of classification and all surfaces described by numbers. Now, as a mathematician I'm seeing a pattern here. There are a bunch of negative ones, positive ones, so there must be other information.
17:06	Leibon: There is, and in fact you've hit on a really important one, especially from a modern mathematical point-of-view, which is the sign of that Euler characteristic tells you something very special about that surface. For example, we have our sphere over here we could imagine as this mushy beach ball, but it has all these geometric realizations, you know, ways we can think of it. But there's one that's really beautiful, and that's the round sphere, the one with all the symmetry and a very beautiful, very uniform object. And that sphericalness of it is captured by the positiveness of that number.
17:36	Leibon: Now, the zero tells us another bit of geometry, which is namely that it's flat like Euclidean plane. And we've experienced this with our asteroids.
17:44	Leibon: So this zeroness tells you you look like the Euclidean plane. And so we've covered, well, in total just a couple of our surfaces, in the orientable case just two; the rest of them are negative. Do they have an equally beautiful geometry, and it turns out they do much better. They have what I think of as the most beautiful of all geometries, which is hyperbolic geometry.
18:03	Leibon: The hyperbolic plane is this amazing saddle space where every point looks exactly the same, exactly the same shape saddle. It's this totally uniform

	space.
18:12	Leibon: So this is interconnecting the geometry to the topology. And the main reason we really want to understand this is because in three dimensions, this is the connection. Our classification is about the geometry, and the geometry --
18:22	Rockmore: Right. And you've made this link now. Leibon: That's right. Rockmore: We started off saying we're going to forget about geometry, and now we've actually said, "Well, we couldn't -- in some ways we couldn't get away from it."
18:29	Leibon: We couldn't get away from it, and that was an important lesson for mathematics to learn.
18:33	Rockmore: …So thanks for helping us see like a topologist. Leibon: It was my pleasure, Dan. Rockmore: All right. Terrific. So let's go see if this asteroids guy can make any progress.
18:40	**HOST (V.O.):** **So if we go back to our game of Asteroids...and we turn our flat Euclidean space back into a torus...**
18:46	**HOST (V.O.):** **...we've taken a single right-angle square and created a torus to build my game on.**
18:52	**HOST (V.O.):** **...so what we're seeing on our flat screen when we play the game is one those squares. The geometry is there and it's Euclidian.**
19:00	HOST So with Asteroids, globally we've got the topology of a torus, and locally the geometry is that of a plane.
19:08	HOST We've gone from taking geometry out of the problem to creating an abstraction like a torus that's all about space and shape to give us a way to classify information, and then put geometry back into the problem. And we used the Euler Characteristic to show us which of three geometries we can use. What happens when we go to three dimensions? Can we understand 3-dimensional space just like we can 2- dimensional space? The answer is yes -- but it took several mathematicians an entire century to figure it out...
19:42	HOST

	When mathematicians talk about three-dimensional spaces – or manifolds, they mean something a little different from the usual height, width and depth most of us think about. Now, two-dimensional manifolds, like the surface of a ball, can be visualized as the skin of a three dimensional object - something that we can pick up and hold in our hands. This is a two-dimensional sphere because we can determine any point with just two numbers – the latitude and the longitude. Now, try to think about a three-dimensional sphere. This has to be the skin of a four-dimensional object. Or even a four dimensional sphere. Well, you may be surprised to hear that mathematicians understand four, five, and all higher dimensional spheres, but three-dimensional spheres are still a mystery. Characterizing a three-dimensional sphere would amount to settling the famous Poincare Conjecture, a one hundred-year old challenge, named for the great French mathematician Henri Poincare, and chosen as one of the seven Millennium Problems. It now looks like the Poincare Conjecture has been settled by Russian mathematician Grigori Perelman. In doing so, he may have provided the key to understanding what three-dimensional spaces look like, including the shape of our universe. The shape of our universe is a deep mystery, and Jeffery Weeks is one of the mathematicians who is trying to figure this out...
21:08	Jeff Weeks: You go out into nature, you see the world, you look around, you want to understand what you're seeing. So it's like if you see little points of light in the sky, okay, you call them the stars, but what are they? The same is true with the universe as a whole.
21:29	Weeks: I'm Jeff Weeks. I'm a mathematician. Topology and Geometry are my specialties. Over the past 10 or 15 years, I have had the pleasure of applying geometry and topology to the study of the real universe to test whether our universe is infinite or finite.
21:44	*Ranger:* *We're here in the main telescope room of Goldendale Observatory.*
21:47	Weeks: We study the shape of the universe to satisfy basic human curiosity. We're born in this world, we look out, we want to understand where we are. *Ranger:* *As the Planets orbit the sun…* Weeks: And the mathematics is really a means to an end here, to take what we see, sort of the raw data, looking at the sky, make some sense of it, and understand how the universe is built.
22:11	Weeks: Humans have been thinking about this for over two millennia that the ancient Greeks thought a lot. Some thought the universe was infinite, others thought it was finite but had a boundary. You could just go out to this final sphere and you couldn't go out any further. And it wasn't until more modern times when, in the 19th century, that mathematicians came up with the idea with

	constructing a universe that's finite but has no boundary.
22:37	Weeks: What happens in our own research is we look at different possible shapes for the universe, and then for each one, we make predictions about what we'd expect to see in the sky in each case, and we compare those to have actual observations.
22:51	Weeks: This is just a tremendously exciting time to be studying this question because finally we have some real data that allows us to address it and in fact data that at least very tentatively suggests that the real universe might indeed be finite.
23:08	Weeks: The most interesting data is data that looks at the fluctuations, the ripples in the microwave background. So the microwave background is coming from our horizon sphere, so it's basically as far out as we can possibly see. The largest possible scale we can see on. In an infinite universe, you'd expect to have fluctuations on all scales, so it's kind of like if you take water in your bathtub and slosh it around, you'll get some longer waves, you'll get some shorter waves, you get a lot of little choppy waves. All superimposed on each other.
23:40	Weeks: The same thing's going on with the microwave background, and indeed in an infinite universe, you'd expect fluctuations on small scales, medium scales, and the largest possible scales as well. But when you look at the real universe, you see fluctuations on the small scale, as expected, fluctuations on medium scales, as expected. The surprise is on the largest scales, the fluctuations are almost entirely missing. They're just not there.
24:07	Weeks: The COBE satellite went up in the 1990s and took a very blurry view of our horizon sphere, and this is where people first noticed that the broadest fluctuations seemed to be very weak or missing. Then more recently, NASA sent up the WMAP satellite, which has taken a far more precise view of the microwave sky and again, confirming COBE's observation that the largest fluctuations are indeed missing.
24:35	Weeks: There are two approaches to taking data and comparing that to the different possible models.
24:41	Weeks: You can look at the harmonics, the overtones, and look at the relative strengths of those harmonics. We see these fluctuations, but just like the musical instruments had harmonics, the sky has its harmonics, and in different spaces with different shapes, each one has its characteristic set of harmonics.
25:02	Weeks: ~~A~~ second method is to actually look and see whether part of the horizon sphere seen on one part of the sky matches up with that scene on the other part of the sky. So far, those searches have turned up negative. Meaning people have looked and not found them.

25:17	Weeks: So you want to take a model for a possible shape for space and ask does that correspond to reality? So topology let's you explore what the candidates are and it also lets you explore them in a sensible way.
25:32	Weeks: This research is important because it satisfies humanity's basic curiosity to understand the world we live in. So it's like if you go for a walk in the forest, okay, you see the trees and they're pretty and everything, but if you understand a little how they work, that each leaf is composed of cells and the cells absorb sunlight. Your understanding is much richer and your appreciation of the tree's beauty is much richer. And I think the same is true with the universe. You can look out at the night sky, see some specks of light, it's pretty, but if you understand how big it is, how it's made, the fact that there are galaxies out there, if you understand, you know, if the universe is indeed finite, if you understand that, it makes your appreciation of your universe all the deeper.
26:14	Weeks: It's interaction between the mathematical theory and understanding nature is what really makes it a joy for me.
26:24	HOST Turning a tour around town into a walk on a graph; Having fun by turning a donut into a coffee cup; Taking on the challenges, mysteries, and excitement of trying to see and even feel the shapes of things in higher dimensions; Trying to use all of our powers of exploration of inner space to make sense of the shape of outer space...All of that is topology. Have some fun with it!
26:50	CLOSING CREDITS