MATHematics illuminated

TEXTBOOK

UNIT 3

UNIT 03

HOW BIG IS INFINITY?

TEXTBOOK

UNIT OBJECTIVES

- Ideas of infinity come to light when considering number and geometry, the worlds of the discrete and the continuous.
- Incommensurability is the idea that there is no measurement unit that fits into some two quantities a whole number of times.
- Incommensurability led to the discovery of irrational numbers.
- Irrational numbers have decimal expansions that never end and never repeat.
- Two sets are the same size if their elements can be put into one-to-one correspondence with one another.
- The size of a set is its cardinality.
- There is more than one type of infinity.
- The sets of rational and real numbers are examples of two different sizes of infinity.
- To properly describe the different sizes of infinity, a new definition of number is required.
- Given a set of any size, one can create a larger set by taking the subsets of the original set.

“ ”

It is well known that the man who first made public the theory of irrationals perished in a shipwreck in order that the inexpressible and unimaginable should ever remain veiled. And so the guilty man, who fortuitously touched on and revealed this aspect of living things, was taken to the place where he began and there is forever beaten by the waves.

PROCLUS DIADOCHUS (412 - 485)

“ ”

If you disregard the very simplest cases, there is in all of mathematics not a single infinite series whose sum has been rigorously determined. In other words, the most important parts of mathematics stand without a foundation.

NIELS H. ABEL (1802 - 1829)

SECTION 3.1

INTRODUCTION

From an early age, we have an intuitive sense that there can be no biggest number. As soon as we learn how to add two numbers together, we have at our disposal a mechanism by which we can make any number bigger—just add one! We have a sense of both a process and a set—the set of all numbers—that are infinite, larger than anything in our daily experience. We also learn a hierarchy of numbers: a billion conquers a million, a googol beats a billion, and infinity is the sovereign value, untouchable in its perfection.

What exactly is infinity? Does it really exist? It certainly doesn't play any obvious role in our everyday lives. We are finite, and we live in a finite world. Our lives have definite beginnings and endings, and we measure the time between these two points using discrete, finite, units such as years, minutes, and seconds. Similarly, the physical space in which we live our lives and enact our everyday pursuits is bounded and separated into fundamental units, such as miles and millimeters. Our best bet for grasping some sensory experience of infinity is to gaze toward the heavens on a starry night. It remains an open question, however, whether or not the universe actually extends forever.

The process of adding the number one to another number to make it greater does not make the result infinite—it merely makes another, greater, finite number. The Greeks called a quantity or a collection "potentially infinite" if, given any finite example of that quantity, a "larger" example could always be found. In this respect, a line segment (a "collection" of points) is potentially infinite, because it can always be made longer, and the set of counting numbers is potentially infinite, because from one counting number, we can always construct a greater one. To conceive a quantity that is actually infinite, however, is mind-bending and, in many ways, perplexing. What happens if we add the number one to an actually infinite number? It's already infinite—does adding to it make it greater? How could one possibly have something "more" than infinity?

Such a concept is often called "actual infinity," and it is much more problematic than "potential infinity." It defies intuition, forcing us to rely on logic to explore the defining aspects of the concept. The idea of actual infinity has been disturbing to mathematicians since at least the time of the Greeks. At times, it seems to be more an invention of the human imagination than anything real, and, ideally, mathematics should be the language that describes reality, not fantasy.

SECTION 3.1

INTRODUCTION
CONTINUED

Despite its nebulous reality, the concept of infinity has long teased at mathematicians' minds. Around 500 BC it manifested in the form of incommensurable quantities, a concept akin to heresy in the view of many, particularly the followers of Pythagoras. At almost the same time, paradoxes posed by the philosopher Zeno showed that infinity was a difficult concept for the human mind to comprehend. It was at this point that the Greeks reluctantly accepted the use of infinity in mathematics, but they left the challenge of understanding it to philosophers and priests.

This view persisted for centuries. Infinity was a tool that could be used in mathematics, even if it was not well understood. As it turned out, infinity proved to be an indispensable tool for 16th- and 17th-century mathematicians seeking to use mathematical concepts to describe real-world physical phenomena, This is most evident in the field of calculus. In order for calculus to work (and we assume that it does work, because it describes the physical world superbly), we have to believe that actual infinity exists—that is, we have to believe at least that an infinite process can have a finite result. So, the concept of infinity proved useful then in much the same way that a modern cell phone does now; we certainly don't have to know how it works in order to make use of it.

Mathematics, however, is supposed to be based on rock-solid, well-understood principles. As the tower of mathematics grew larger and more intricate, with each new idea depending on the validity of those that came before it, mathematicians began to double-check the foundational principles. They were concerned that it might be a bad idea to base large parts of our understanding of the world on a concept, infinity, that we fundamentally do not understand. Enter Georg Cantor. Cantor sought to understand mathematically the infinite and the consequences of believing in an actual infinity. He did this by creating the language of sets, which are just collections of objects, such as numbers. In doing so, he had to redefine what a number really is. Through some of the most creative and ingenious mathematics ever done, Cantor showed, contrary to intuition, that there can be different sizes of infinity. His polarizing results generated much controversy that, to this day, is not completely resolved.

In this unit we will explore infinity by first looking at rational numbers and some of their properties. We will then see how incommensurable quantities and irrational numbers suggest that infinity is at work in the number system. Through Zeno's paradoxes, we will catch a glimpse of how difficult infinity can be to understand. From there we will look at sets of numbers and re-learn how to count in a way that will enable us to approach the concept of infinity. With these tools in hand, we will get a sense of the universe of infinities that Cantor discovered, culminating in Cantor's Theorem, one of the most counter-intuitive ideas in mathematics.

SECTION 3.2

RATIONAL NUMBERS

- Common Denominator
- The Ever-Expanding Decimal
- Number vs. Magnitude

COMMON DENOMINATOR

- Rational numbers arise from the attempt to measure all quantities with a common unit of measure.

The pursuit of infinity begins with an examination of the idea of number, or quantity. Numbers originally were tools used to quantify groups of objects or to measure real things, such as the length of a pole or the weight of a piece of cheese. Measuring something requires some fundamental unit that can be used as a basis of comparison. Some things, such as rope or time, can be measured, or quantified, using a variety of units. In measuring a length of rope, for example, we might express the result as either "5 feet" or "60 inches." The length of time from one Monday to the next Monday is commonly called a "week," but we could just as easily—and correctly—call it seven "days," 168 "hours," 10,080 "minutes," or 604,800 "seconds." These number expressions all represent the same length of time and, thus, are interchangeable. Converting any one of these equivalent values into another simply requires multiplying or dividing by some whole number. For example, 10,080 minutes is 168 hours times 60. In fact, every one of the above measurements could be converted into seconds by multiplying by appropriate whole number values.

Now, we can take any two quantities and ask a similar question: can we find a common unit of measurement that fits a whole number of times into both? Take 8 and 6, for example. This is straightforward; both 8 and 6 are whole numbers, so we can use whole units to measure them both. What about $\frac{1}{2}$ and $\frac{3}{8}$? Here, each of the quantities uses different base units, namely halves and eighths, and comparing them would make little sense, because they are different things. We can find a common unit of comparison, however, by recognizing that $\frac{1}{2}$ is the same as $\frac{4}{8}$.

So, we found a common denominator of 8, which implies that both of the fractions could be expressed as multiples of the same unit, "one eighth." In some sense, we have redefined our basic unit of measurement, or fundamental

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RATIONAL NUMBERS
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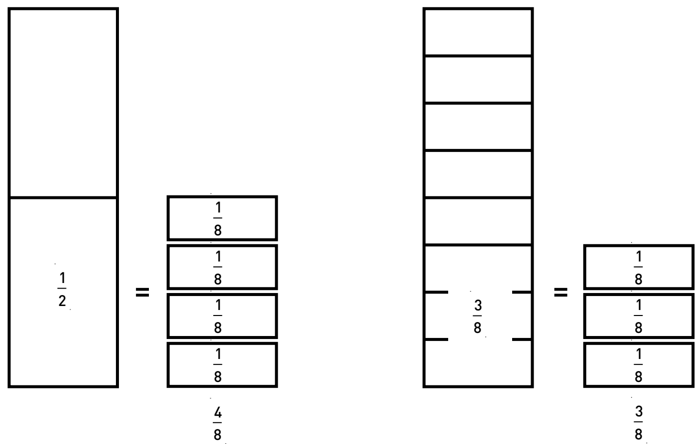
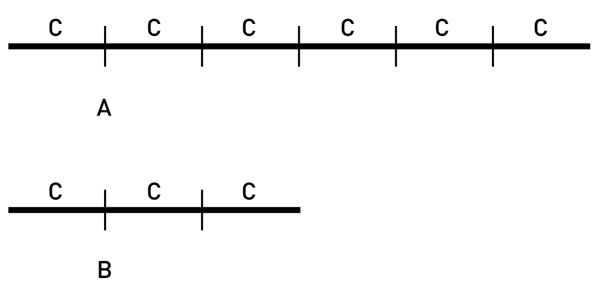
piece, to be $\frac{1}{8}$ of the original piece, thereby transforming the original fractions into easily-compared multiples of the same fundamental unit.



If two numbers can be expressed as whole-number multiples of some common unit, of whatever size, they are in some sense "co-measurable" in that we can measure both using the same ruler. The proper mathematical term for "co-measurable" is "commensurable." One way to think about this is that two lengths are commensurable if there is a basic unit of measure that fits into both of them a whole number of times. If we were to cut some length of rope into two pieces of lengths a and b, there would be some third length, c, such that $a = mc$ and $b = nc$. In other words, these two numbers could be expressed as multiples of some common unit.



Using a little algebra, we can confirm that the ratio of magnitudes of our two commensurable quantities is equal to a ratio of whole numbers:

$$\frac{a}{b} = \frac{(mc)}{(nc)}$$

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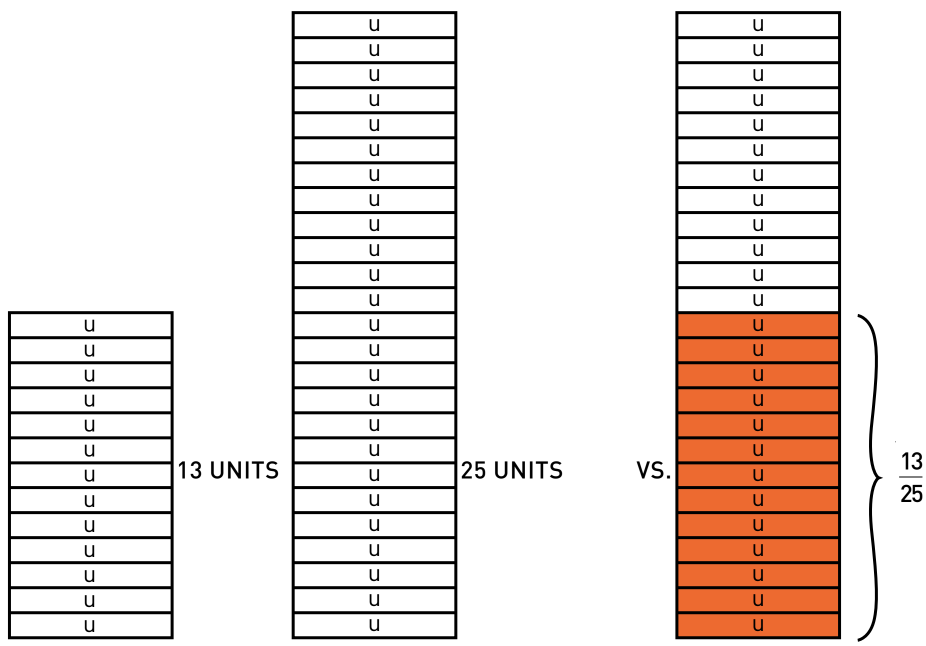
RATIONAL NUMBERS
CONTINUED

Canceling the common factor of *c* yields the equation:

$$\frac{(mc)}{(nc)} = \frac{m}{n}$$

Numbers that can be expressed as ratios of two whole numbers are called rational numbers. This idea of "creating" numbers that may not relate to any observable value in the real world is fairly modern. Although today we are comfortable speaking of a number such as $\frac{13}{25}$ as a concept that "exists" in its own right, the ancient Greeks generally took care to phrase things only in terms of geometric quantities--those that exist in the physical world. For instance, they might have spoken of two lengths of rope, one that could be described as 13 measures of a certain unit, and the other 25 units of that same measure, but they would not necessarily speak of the shorter length being $\frac{13}{25}$ of the other.



On the other hand, today we are comfortable saying, for example, that a length of string is $\frac{13}{25}$ of an inch long. In doing so, we are saying that it is commensurable with a piece of string one inch long, the fundamental unit of comparison being $\frac{1}{25}$.

The modern and ancient views of rational numbers are intimately linked, but it is important to remember that the Greeks thought of commensurability in terms of whole units. The Pythagoreans, the followers of Pythagoras of Samos

SECTION 3.2

RATIONAL NUMBERS
CONTINUED

in 6th century BC Greece, held sacred the idea that the first principle underlying everything is "arithmos," the intrinsic properties of whole numbers and their ratios. It is certainly a tidy idea that whole numbers, or ratios of them, are all that is required to describe the world mathematically. It is thought that this belief had origins in both the study of figurate numbers and the recognition that strings or hammers of commensurable length sounded harmonious when played or struck together.

THE EVER-EXPANDING DECIMAL

- Rational numbers can be expressed as decimals that repeat to infinity.

In any ratio of two whole numbers, expressed as a fraction, we can interpret the first (top) number to be the "counter," or numerator—that which indicates how many pieces—and the second (bottom) number to be the "namer," or denominator—that which indicates the size of each piece.

In modern arithmetic, we use a base-10 system to count, or evaluate, things. Large quantities are generally represented in terms of ones, tens, hundreds and the like, whereas small quantities are more easily represented in terms of tenths, hundredths, thousandths, and so on. Although the Greeks did not use a base-10, or decimal, number system, it is illuminating to see how rational numbers behave when expressed as decimals.

For example, we can interpret the number 423 as four 100s, two 10s and 3 units (or 1s), and the value 0.423 as four ($\frac{1}{10}$)s, two ($\frac{1}{100}$)s and three ($\frac{1}{1000}$)s. In such a decimal system it is necessary to think of all quantities in terms of units of tens, tenths and their powers. Thus, $\frac{1}{2}$, for instance, must be interpreted as $\frac{5}{10}$, to be written as 0.5.

The question of whether or not 0.5, or $\frac{5}{10}$, represents the same quantity as $\frac{1}{2}$ deserves a bit of thought, however, because it highlights a subtle difficulty with our understanding of rational quantities (and maybe with our understanding of "number" itself). To explain it, let's return to the Greek point of view of commensurability. Recall that two lengths, a and b, are commensurable if there exists a common unit of measure, u, such that each length can be generated by taking u a whole number of times: a = mu and b = nu. Applying algebra, it is easy to confirm that the ratio $\frac{a}{b}$ equals the ratio of whole numbers $\frac{m}{n}$. What would happen, though, if we worked with a smaller unit, v, that fits five times into u (that is, u = 5v)? Then, we would have: a = 5mv and b = 5nv, and the ratio

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RATIONAL NUMBERS
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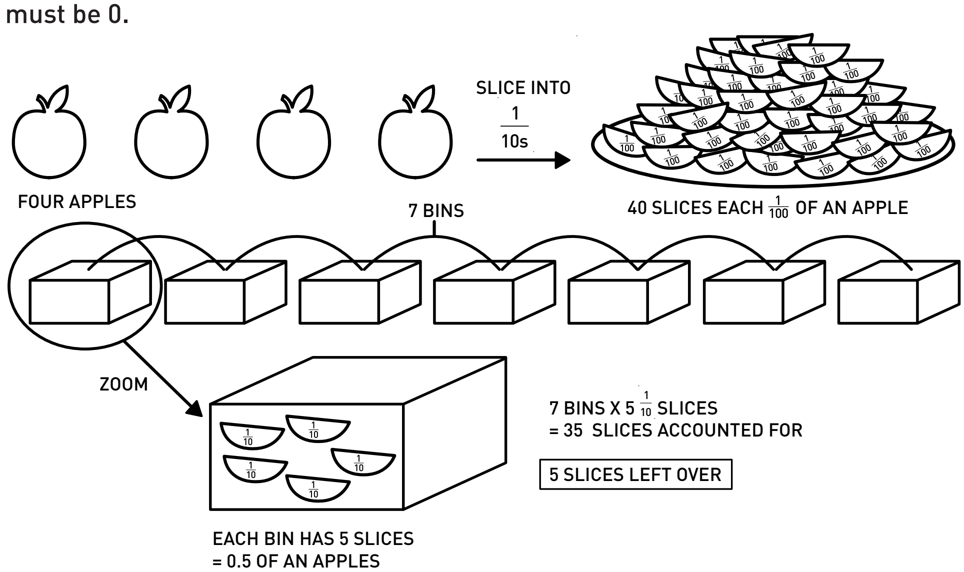
$\frac{a}{b}$ still would be equal to the ratio of whole numbers $\frac{5m}{5n}$, or $\frac{m}{n}$. This shows that a rational number is not simply a ratio of any two specific whole numbers, but rather, represents a collection of "equivalent" whole-number ratios. A consequence of all of this is that our modern notion of a rational number is, in itself, somewhat abstract and troublesome to comprehend. Putting philosophical woes aside for the present, we have at least seen that $\frac{1}{2}$ and $\frac{5}{10}$ are different representations of the same ratio.

Many find it useful to view rational quantities as answers to division problems. For example, sharing one apple equally between two students results in each student receiving half of an apple. Dividing two apples equally among three bins yields $\frac{2}{3}$ of an apple per bin. Thankfully, each equivalent representation of a rational number, interpreted as a division problem, yields the same physical result: dividing four apples among six bins, and 10 apples among 15 bins, and 200 apples among 300 bins, all yield the same result as dividing two apples among three bins.

We will use this division model to our advantage as we convert fractions into decimal representations. For example, to write $\frac{4}{7}$ as a decimal number, we can think of the process of dividing four things, such as apples, among seven bins. Our decimal representation is then the number of apples in each bin, with one whole apple being our fundamental unit. Because we are dividing only four apples equally into seven bins, we realize that each bin must receive less than one whole apple, so the value in the 1's place of our decimal-expansion number must be 0.



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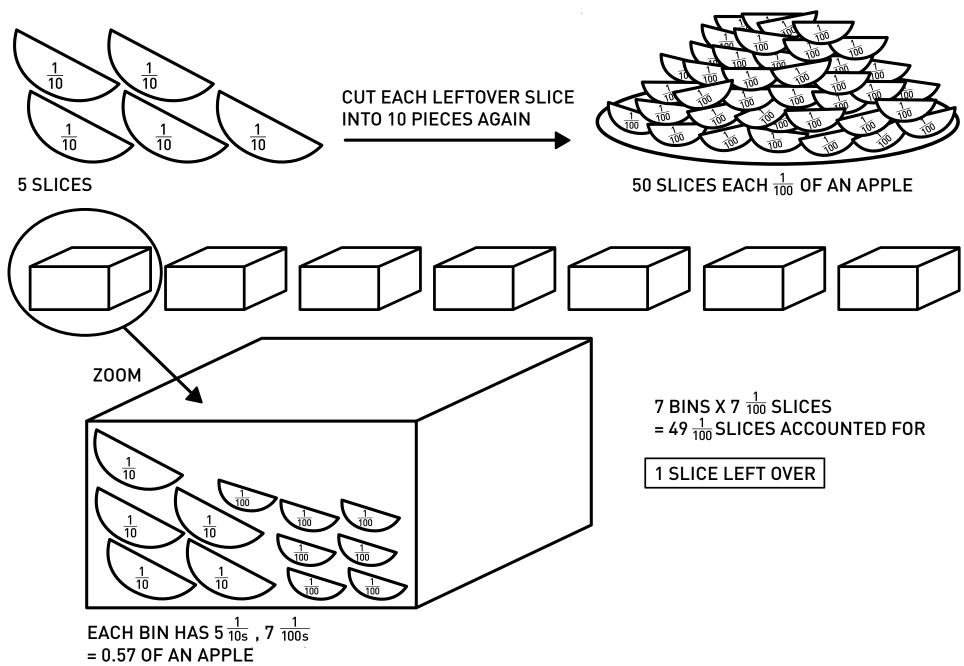
How do we actually envision this process, though? What would we actually do to the four apples to achieve the equal distribution into the bins? We could begin by cutting each apple up into ten equal pieces, which would give us 40 slices, each being a tenth the size of a whole apple. If we were then to apportion these slices equally into the seven bins, each bin would receive five slices (or five-tenths of an apple) with five slices left over. Note that the content of each bin after this initial distribution is represented by the decimal 0.5.



If we repeat this process, dividing the five leftover tenths each into ten equal pieces, we would have 50 slender slices, each being a hundredth of a whole apple. Apportioning these slices equally among the bins would mean that each bin receives seven slices (or $\frac{7}{100}$ of an apple), with one slice left over. The accumulated total in each bin can now be represented by the decimal 0.57.

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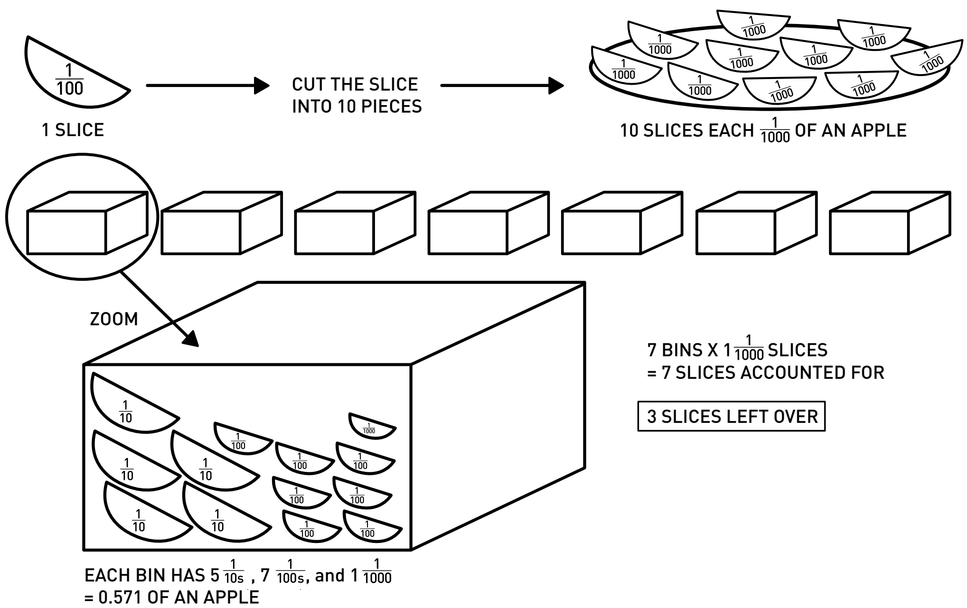
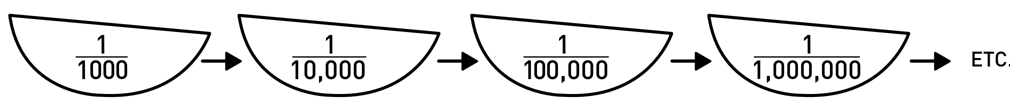
RATIONAL NUMBERS
CONTINUED



If we take the one leftover slice and cut it into ten equal pieces, we will create slices that are each just a thousandth of a whole apple. With equal distribution, each bin receives just one of these slices, and three are left over. The total amount of whole apple now in each bin can be represented by the decimal 0.571.



We can continue this process of dividing each leftover slice into ten pieces, placing an equal number of slices into each of the seven bins, and then dividing the leftovers again, indefinitely. In this particular example, we would soon find that the number sequence repeats itself after six decimal places so that the decimal representation of $\frac{4}{7}$ is 0.571428571428.... Why must the decimal repeat? In our example, there are only six choices for non-terminating remainders (i.e., 1, 2, 3, 4, 5, and 6). Note that a remainder of zero would end the division process and create a terminating decimal. In the absence of termination, one of the remainder values must reoccur, thereby beginning a repeating sequence of numbers.

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RATIONAL NUMBERS
CONTINUED

Note that this expansion never ceases, continuing for as long as we care to continue the division process. This is somewhat reminiscent of the potential infinity we talked about in the introduction to this unit. We can always determine another decimal place value, but after recognizing the repeating pattern, we don't need to.

There is nothing special about the ratio values 4 and 7 in this example. The logic of breaking up leftovers into ten equal pieces and distributing those pieces equally holds for whichever two numbers we choose. In this way, any rational number can be written as a repeating decimal. Even fractions that can be represented by "terminating" decimals, such as $\frac{1}{2}$, can be thought of as repeating, if we recognize that $\frac{1}{2} = 0.5 = 0.5000000....$

Conversely, any repeating decimal can be shown to be a ratio of whole numbers. Consider, for example, the decimal 0.4444.... If we let $x = 0.4444...$, we are saying that x consists of four $\frac{1}{10}$s, four $\frac{1}{100}$s, four $\frac{1}{1,000}$s, and so on. Ten times this value ($10x$) would then be four units (1s), four $\frac{1}{10}$s, four $\frac{1}{1,000}$s, and so on—or, more concisely, $10x = 4.4444....$

With the decimal values of both x and $10x$ established, we can construct this calculation:

$$\begin{array}{rl} 10x = & 4.4444... \\ -\ x = & -0.4444... \\ \hline 9x = & 4 \end{array}$$

Solving the resulting equation for x, we get:

$$x = \frac{4}{9}$$

Notice that this works because every 4 to the right of the decimal point in the number 4.4444... matches up with a 4 to the right of the decimal point in the number 0.4444.... When the two numbers are subtracted, all these 4s completely cancel out.

This method of converting a repeating decimal into a fraction also works for

SECTION 3.2

RATIONAL NUMBERS
CONTINUED

decimals that have longer repetition sequences, such as 0.325325325....
Let x = 0.325...
Then 1,000x = 325.325...

$$\begin{array}{rrl} & 1{,}000x = & 325.325\ldots \\ - & x = & -\ 0.325\ldots \\ \hline & 999x = & 325 \end{array}$$

$$x = \frac{325}{999}$$

In the above example, both 0.325... and 325.325... exhibit an infinite decimal expansion, yet we can cancel all the digits to the right of the decimal point because it is plain to see that each decimal digit in 325.325...matches up with an equivalent decimal digit in 0.325...; leaving only the whole number 325 after subtracting the two quantities. This idea of establishing a one-to-one correspondence among the decimal digits provides a glimpse of how we might think mathematically about infinity that will be of supreme importance later on in this unit.

NUMBER VS. MAGNITUDE

- In the mathematics of early Greece, there was a strong distinction between discrete and continuous measurement.
- Number refers to a discrete collection of atom-like units.
- Magnitude refers to something that is continuous and that can be infinitely subdivided.
- Rational numbers can be expressed as decimals that repeat to infinity.

Early Greek mathematicians divided mathematics into the study of number, or multitude, and the study of geometry, or magnitude. The multitude concept presented numbers as collections of discrete units, rather like indivisible atoms. Magnitudes, on the other hand, are continuous and infinitely divisible. Because length is a magnitude, a line segment can be divided as many times as one likes.

The Pythagoreans believed that magnitudes could always be measured using whole numbers, which would imply that lengths are not infinitely divisible. Other schools, such as the followers of Parmenides, known as the Eleatics, believed in the infinite divisibility of magnitudes.

Parmenides taught that true "being" is unity, static, and unchangeable. This

SECTION 3.2

RATIONAL NUMBERS
CONTINUED

is similar to the idea that "all is one," which implies that concepts such as multiplicity and motion are illusions. If everything is part of the same thing, then there are no "multiple" things and, consequently, no motion, which is the change in position of one thing relative to another. Pythagoreans believed in multitude and motion perhaps because these concepts are intuitive, part of collective common experience. A consequence of the Pythagorean notion of multiplicity is that magnitudes should be commensurable. To the Pythagoreans, the idea that between any two quantities in nature there exists a common unit of measure, a common denominator, may have been comforting. It perhaps suggested that the rational mind can always find a solid basis for comparison, and does not have to rely on guesswork to say definite things about reality.

It would be easy to dismiss the Eleatic view, if it were not for the arguments of one of Parmenides' most famous pupils, Zeno. As we shall see, Zeno argued against the Pythagorean notions of multiplicity and motion, using infinity to show contradictions in this view. Prior to Zeno, however, problems with the Pythagorean viewpoint arose from within their own ranks in the form of an independent thinker by the name of Hipassus of Metapontum. Hipassus showed that magnitudes are not always commensurable, an idea that upset his peers to such a degree that, as the legend goes, he was drowned for his heresy. In the next section, we shall examine the idea and consequences of incommensurable magnitudes.

SECTION 3.3

INCOMMENSURABILITY AND IRRATIONALITY

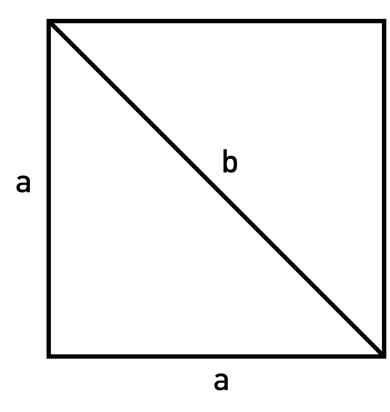
- Odds and Evens
- The Infinite Chase

ODDS AND EVENS

- The side and the diagonal of a square are incommensurable.

To recap, the Greek concept of magnitude was somewhat tied to what a person could measure in the real world. The Pythagoreans believed that all magnitudes in nature could be represented through arithmos, the intrinsic properties of whole numbers. This means that for any two magnitudes, one should always be able to find a fundamental unit that fits some whole number of times into each of them (i.e., a unit whose magnitude is a whole number factor of each of the original magnitudes)—an idea known as commensurabilty. Hipassus argued against this idea by demonstrating that for some magnitudes this simply isn't the case—they are incommensurable. Although his original argument is lost to the ages, the following proof, which uses algebraic notions that would have been unfamiliar to the Greeks, gives a sense of the discovery that changed Greek mathematics forever.

Let's imagine a square with a side of length *a* and diagonal of length *b*.



If these lengths are commensurable, as Pythagoras and his followers believed (without proof), then there is a common unit u such that $a = mu$ and $b = nu$ for some whole numbers *m* and *n*. We can assume that *m* and *n* are not both even (for if they were, it would indicate that the common unit could instead be $\frac{1}{2}u$, and we would simply make that adjustment). So, we can safely assume that at least one of these numbers is odd.

SECTION 3.3

INCOMMENSURABILITY AND IRRATIONALITY
CONTINUED

Applying Pythagoras' theorem to the triangle formed in the square, we have:

$a^2 + a^2 = b^2$

That is,

$2a^2 = b^2$

or, substituting our common unit expressions for the two lengths,

$2m^2u^2 = n^2u^2$

We know that our common unit, u can't be zero, so we can cancel the u^2 term from both sides of the equation, leaving:

$2m^2 = n^2$

Obviously, n^2 is even, because it is equal to some number, m^2, multiplied by two. If n^2 is even, then n must be even also (if n were an odd number, then n^2 would be odd). We can express the even number n as two times some number.

$n = 2w$

Substituting this expression for n into the preceding equation gives us:

$(2w)^2 = 2m^2$

$4w^2 = 2m^2$

$m^2 = 2w^2$

This reveals that m^2 is a multiple of two, that is, an even number. Consequently, as we reasoned before, m must also be even, and we can write:

$m = 2h$

Now we have found a contradiction! Remember, we assumed at the beginning that either *m* or *n* was odd, yet we have just shown that both have to be even. This logical contradiction proves that there is no common unit, *u*, that fits a whole number of times into both *a* and *b*—therefore, *a* and *b*, the lengths of the side and diagonal of a square, are incommensurable.

THE INFINITE CHASE

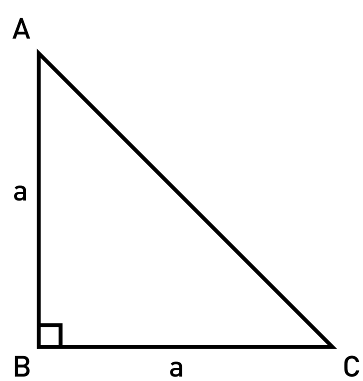
- Incommensurable quantities are not rationally related, because this logically leads to an infinite regress.

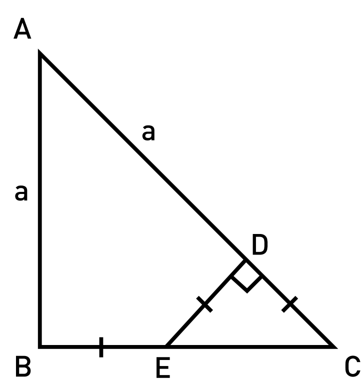
What does incommensurability have to do with infinity? A contemporary of Hipassus, Theodorus of Cyrene, proved the incommensurability of the side and diagonal of a square by showing that no matter how small of a unit one uses to measure the side and the diagonal, it will never fit a whole number of times into both. In fact, selecting smaller and smaller units merely leads to an infinite regression of triangles. Theodorus' approach is illuminating in that it is more in line with how the Greeks thought about mathematics than the previous demonstration of incommensurability.

To get a sense of Theodorus' proof, let's again focus on the isosceles right (also commonly called a "45°-45°-90°") triangle formed by two sides and the connecting diagonal of a square. Designating this as triangle ABC, with legs of length *a* and hypotenuse length *b*, let's once more assume that there is a fundamental unit of measurement capable of representing the lengths of both a side and the hypotenuse in whole number multiples; that is:

$a = mu$ and $b = nu$

Along the hypotenuse of the right triangle, we can measure a length equal to the side's length and construct a new 45°-45°-90° triangle CDE as shown:



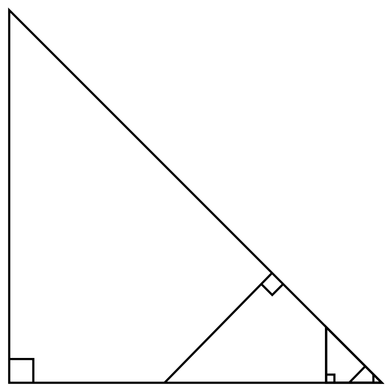


SECTION 3.3

INCOMMENSURABILITY AND IRRATIONALITY
CONTINUED

Without too much difficulty, we can show that all three segments, BE, ED, and DC are congruent. (We won't go through the proof, but you would begin by constructing a line from A to E and showing that the two triangles ABE and ADE are congruent.) and that each of these lengths is *b-a*, again a whole number of copies of *u*.

Thus, from any 45°-45°-90° triangle with sides whose measure is a multiple of *u*, we can construct a smaller 45°-45°-90° triangle with sides whose measure is also a multiple of *u*. We can keep doing this for a number of iterations.



Eventually, however, we will obtain a 45°-45°-90° triangle so small that the length of each of its sides is less than *u*, which can't be—*u* was supposed to be the fundamental unit! We might be tempted to think that perhaps *u* was too big to be the fundamental unit. Using a smaller unit, however, would only delay the inevitable fact that at some point we will reach a triangle with sides whose lengths are shorter than our fundamental unit. Choosing ever smaller units leads to ever smaller "terminal" triangles for as along as we care to continue the process, another example of potential infinity. Our beginning assumption that there was a common unit of measure leads to an absurdity.

We have seen two different ways of demonstrating that the diagonal of a square is incommensurable with its side length. It is not uncommon today to calculate that if the side length of the square is 1 unit long, then its diagonal is $\sqrt{2}$ units long. The Greeks, themselves, may not have agreed that something such as this is a number. Recall that the Pythagoreans viewed numbers as discrete collections of atom-like units. This view of numbers requires that we have a whole number "counter" to determine the size of the collection and a whole number "namer" to sit in the denominator of the ratio and designate the size of the unit. However, $\sqrt{2}$ poses a problem because it is not useful in this method; it does not allow us to use whole numbers to serve as "counters" and "namers." This concept put the Pythagoreans in a bind, because it demonstrates that the length of the diagonal of a unit square cannot be a number; consequently, "all is not number." If we insist that such a number must exist because it measures a magnitude that actually does exist, then it is clear that we do not know what a

SECTION 3.3

INCOMMENSURABILITY AND IRRATIONALITY
CONTINUED

number really is. We shall return to this problem a bit later in the text.
The incommensurability argument essentially shows that there are no whole numbers m and n such that $\sqrt{2} = \frac{m}{n}$. We call quantities like these, "irrational," and we have seen that their existence is fundamentally linked to a manifestation of infinity (the infinite regress of Theodorus' proof, for instance.) In the previous section, we saw that any rational number can be written as a repeating decimal and vice versa. However, it doesn't take much thought to conceive of a decimal that does not repeat any finite digit sequence and does not end, such as:

0.101001000100001...

Putting aside for a moment the question of whether or not something like this actually exists, we can say at least that this thing cannot be rational, because if it was it would repeat itself, which it is clearly not going to do. Its decimal expansion extends to infinity with no repetitive elements. This brings us to the point that any non-repeating decimal is non-rational, or irrational. It can also be shown that, like the $\sqrt{2}$, any square root of a number that is not a square number, will also be irrational. Values such as $\sqrt{5}$, $\sqrt{7}$, $\sqrt{103}$, etc., are all irrational.

Shortly after Hipassus made his arguments for incommensurability, which would lead to the discovery of irrational quantities, an Eleatic philosopher, Zeno, would also show the absurdity of a world in which there were fundamental smallest units of space and time. Recall that the Eleatics held beliefs somewhat diametrically opposed to those of the Pythagoreans—that multiplicity, the idea that the universe is composed of fundamental parts—is ridiculous. They believed in continuous magnitudes in which any perceived boundaries were illusions. This idea is somewhat similar to the concept that "all is one." Similarly to Hipassus' argument for incommensurable magnitudes, Zeno would show that treating a line as a multitude of individual points was philosophically contradictory. These ideas would force thinkers to confront notions of actual infinity—an infinity contained in a limited space—which would prove to be both a powerful concept and a troublesome idea in mathematics.

SECTION 3.4

ZENO'S PARADOXES

- Zeno vs. the Multitude
- You Can't Catch Up. You Can't Move. You Can't Even Start.
- Limits

ZENO VS. THE MULTITUDE

- The first of Zeno's arguments shows that considering a line segment to be a collection of points is contradictory.

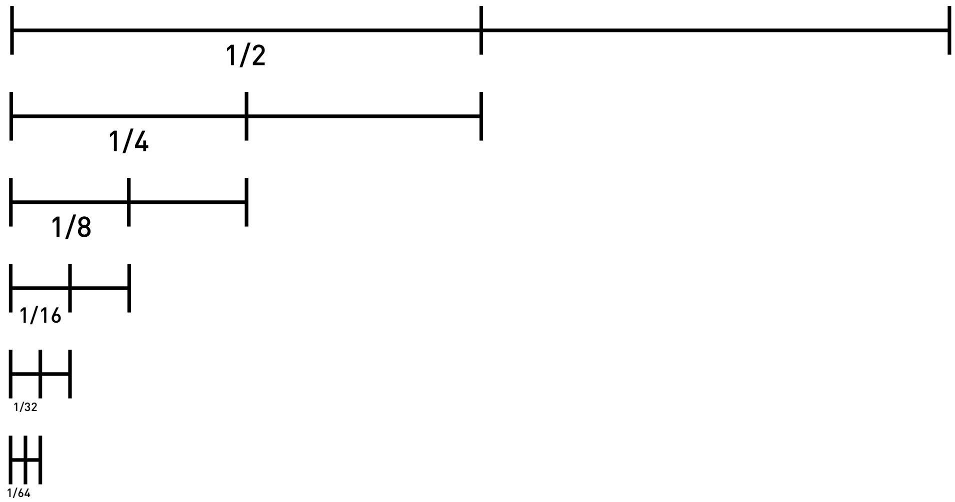
Many early Greeks, particularly the followers of Pythagoras, were fascinated by the idea that whole numbers and their properties provided the first principles upon which all else could be built. Numbers, to the Pythagoreans, were discrete building blocks, like atoms. One of their basic assumptions was that there was always some indivisible unit that could be used to compare any two quantities in nature.

The Eleatic philosopher Zeno proposed a series of philosophical challenges to the notions of multiplicity and motion that demolished the idea of fundamental units of both space and time. That these arguments are paradoxical is due, in large part, to the role of infinity.

Let's first look at Zeno's arguments against multiplicity. In the Pythagorean view, all things in nature could be measured as multiples of a standard unit. For instance, they viewed a line as a collection of discrete points. An infinite line would be an infinite collection of points, but only in the sense of potential infinity, because it was impossible for any person to create a real infinity. A bounded line, a line segment, would, therefore, be constructed of a finite number of points. Zeno, representing the views of the Eleatic school, argued against this view by pointing out that a line segment of any given length can always be bisected, or cut in half. Such a division creates two line segments, each of which can be bisected again, and again, ad infinitum.

SECTION 3.4

ZENO'S PARADOXES
CONTINUED



To a Pythagorean, it was perfectly acceptable to think of an indivisible unit, an "atom," with which magnitude could be "built." Hipassus' argument against commensurability complicated this view somewhat. Hipassus showed that there could be no fundamental common unit between the side and diagonal of a square. As we saw in Theodorus' proof using triangles, this idea of incommensurability implies that a magnitude can be divided as many times as one wishes. Accepting the notion that a magnitude, such as a line segment, can be infinitely divided, or bisected, leads ultimately to the conclusion that any fundamental, atom-like unit must have zero length. This creates a paradox: how can one construct a line segment out of pieces that have no length? One can add zero to zero as many times as one likes and the result will always be zero.

The Eleatic view that a line segment can be infinitely bisected requires that the segment be a continuum with no firm boundaries between one location and the next. The Pythagorean view is based on the concept of discrete parts. Hipassus and Theodorus argued against the Pythagorean view, but Zeno presented a series of four situations that undercut both views. Zeno's paradoxes, although primarily constructed to refute the idea that motion is real, simultaneously manage to argue for and against continuous space (and time), invoking infinity and the absurdities that so often accompany it.

YOU CAN'T GET THERE FROM HERE

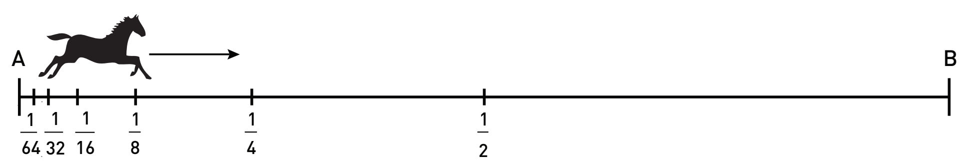
- Zeno's paradoxes of motion are perhaps his most famous and extend his arguments to consider the absurdities of both discrete and continuous space and time.

SECTION 3.4

ZENO'S PARADOXES
CONTINUED

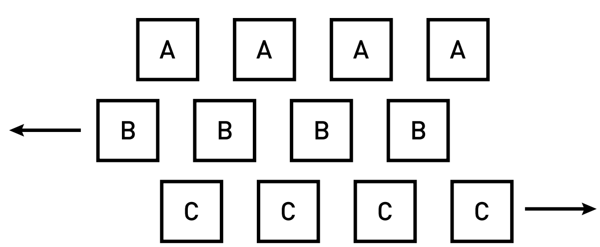
Zeno's most famous arguments have to do with both time and space. He showed that viewing space as a multitude of points and time as a multitude of discrete "moments" forces us to believe that motion is an illusion. Common sense argues against this view, but common sense is informed by our senses, which could, in the view of a philosopher in love with rational deduction, be deceiving us. Zeno presented four arguments against motion: The Dichotomy, The Arrow, Achilles and the Tortoise, and the Stade. Let's look at two of these, the first an argument against continuous space, the second an argument against discrete space and time.

The Dichotomy: Space Cannot Be Continuous



The Dichotomy is very similar to the bisecting line argument we saw in the prior section. In Zeno's example, a horse is trying to traverse the distance from point A to point B. Before it can reach point B, it obviously must first cover half the distance. Before it can cover half the distance, it just as surely must cover a quarter of the distance, and so on. If space is composed of a multitude of points, it must cover an infinite number of these points in a finite time, which is contradictory. Hence, by this line of reasoning, the horse can never make it from point A to point B.

The Stade: Space and Time Cannot Be Discrete



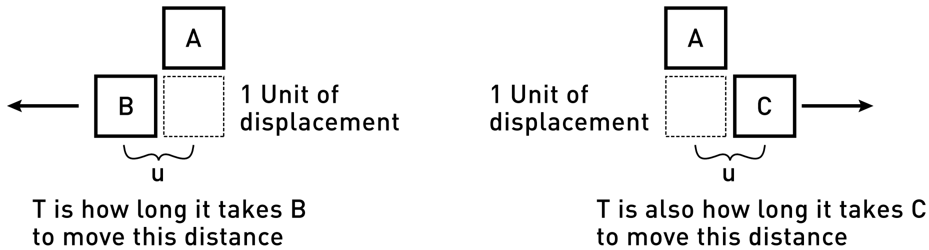
Picture Of Three Four-Box Sequences

This last of Zeno's arguments is more easily understood in a modern example. Suppose that there are three trains, each composed of cars of equal size. Train A is at rest; train B is moving to the left relative to train A; and train C is moving to the right relative to both of the other trains and is traveling at the same speed as train B.

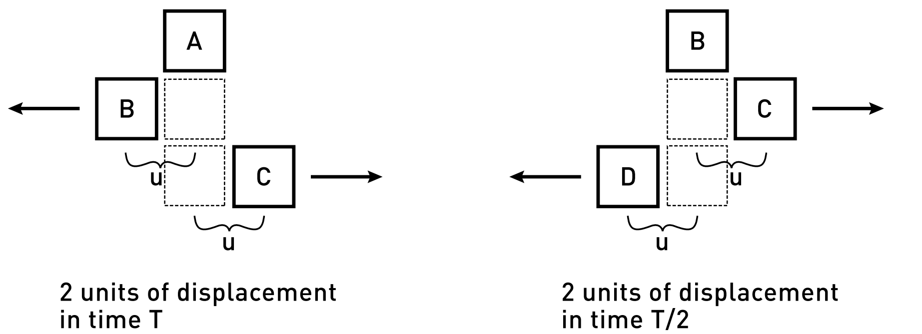
SECTION 3.4

ZENO'S PARADOXES
CONTINUED

Let's say it takes a time, *T*, for one car of train B to pass completely by one car of train A.



Because train C is moving at the same absolute speed as train B, it also takes time *T* for one car of train C to pass one car of train A.



How far do trains B and C move relative to each other in the given time, *T*? Because train B moves one car to the left and train C moves one car to the right, they move two whole cars relative to each other. On the basis of this reasoning, we would be perfectly justified in defining a new smallest unit of time, $\frac{T}{2}$, as the time it takes for train C to move one car relative to train B. This effectively treats train B as being at rest, and we could imagine a new train, train D, and repeat the argument ad infinitum.

The point here is that it is contradictory to imagine time as a series of discrete moments, because those moments can be infinitely subdivided.

LIMITS

- There are multiple ways to resolve Zeno's paradoxes, although many have shortcomings.
- The standard mathematical resolution uses the idea that a sum of infinite, decreasing, quantities can be considered finite.
- The idea of a limiting value to an infinite process is at the heart of calculus.

SECTION 3.4

ZENO'S PARADOXES
CONTINUED

Philosophers and scientists throughout the centuries attempted to resolve Zeno's paradoxes by a variety of arguments. Some denied that space and time exist in any meaningful sense. Some asserted that space and time are not, in fact, infinitely divisible, and moved on. Others used the paradoxes as evidence that our ability to reason is itself contradictory. Still others regarded the distinction between the many and the one to be false, a concept reminiscent of the Eleatic world view that helped spawn the paradoxes in the first place.

Whatever the putative resolutions, it would be a stretch to call any of them mathematical. Mathematicians after Zeno had to accept the existence of actual infinity, even though it does not make intuitive sense. For example, to resolve the paradox of the The Dichotomy, we can look to the convergence of a geometric series, $1 + x + x^2 + x^3$.... It is not hard to show that a general geometric series converges to $\frac{1}{(1-x)}$ as long as $|x|$ is less than 1. To do this requires that we examine the behavior of the series as it approaches infinity. Note that a general geometric series begins with 1, so if $x = \frac{1}{2}$, the sum of the series is then 2.

$$\frac{1}{\left(1-\frac{1}{2}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

The Dichotomy paradox essentially presents an infinite sum of terms of decreasing size $\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)$, which we can recognize to be $\left(\frac{1}{2}\right)^1+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+$ However, unlike a general geometric series, the series implied by The Dichotomy does not start with 1. Consequently, the sum of The Dichotomy series is actually $\frac{1}{(1-x)}-1$, which, with $x = \frac{1}{2}$, equals 1. In other words, the horse makes it from point A to point B.

So, infinity became a tool that could be used, as long as one didn't look too closely at exactly how it worked. Mathematicians came to accept that one could indeed have a finite limit to an infinite sum. This concept made it possible to arrive at a finite magnitude by summing an infinite number of infinitely small pieces. Such pieces, which became known as infinitesimals, have, in some disturbingly vague sense, arbitrarily small but non-zero magnitudes. The great Newton, one of the fathers of the calculus, the revolutionary new theory of the 1600s that described motion both in the heavens and on Earth, at first based his ideas on these troublesome infinitesimals. It wasn't until the 1800s that Augustine Cauchy turned matters around and developed a sound base for the subject by speaking of limits. This had a profound effect on the

SECTION 3.4

ZENO'S PARADOXES
CONTINUED

concept of "number," for Cauchy also found a consistent way to give meaning to irrational quantities, essentially defining them as limits of sequences of rational quantities. For example, let's return to the mysterious quantity 0.101001000100001000.... Cauchy would define this as the limit of the sequence of rationals, 0.1, 0.101, 0.101001, 0.1010010001, ... This shift of perspective represented a marrying, of sorts, of the potential and the actual infinite, and it brought some logic to the concepts of the infinity of irrationals and the infinite sums that arise in calculus.

As calculus began to assume a larger and larger role in both math and science, the need to understand infinity became greater. This quest for understanding ultimately required a shift in thinking, away from looking at whole numbers and magnitudes, toward thinking about sets. In the next section, we will see some of the fundamental ideas in this new way of thinking.

SECTION 3.5

RE-LEARNING TO COUNT

- What Is a Number Anyway?
- Counting to Infinity

WHAT IS A NUMBER ANYWAY?

- To understand infinity, we need a new way to think about what a number is.

After the dual assault by Hipassus and Zeno, mathematicians were forced to accept a world that admits both the discrete and the continuous, the rational and the irrational. We consider rational numbers to be discrete quantities of fundamental units, such as "three"-"sixths" (understood as the quantity three of the fundamental unit one-sixth) or "twenty-five"-"hundredths." Irrational numbers are trickier, requiring an infinite number of non-repeating digits to be expressed in decimal form. The fact that both of these types of things count as "numbers" can be somewhat puzzling. Nevertheless, Cauchy and some of his contemporaries had shown that irrationals, as represented by non-repeating, non-terminating decimals, were essential building blocks for calculus and associated areas of mathematics. Moreover, these innovative mathematicians extended the traditional rules of arithmetic to these number newcomers in such a seamless way that it became clear that the irrationals deserved to be considered numbers every bit as much as their rational predecessors.

At this stage of the development of mathematical thought, the idea of number had been extended from the counting numbers (the naturals) to the rationals (by way of ratios of counting numbers) and on to irrationals (by way of infinite sequences of rationals). All of these numbers, rational and irrational together, formed a large set that came to be called the "real numbers." At each step of this categorizing process, the set of "acceptable" numbers had been enlarged—or had it? Were these new sets really any bigger than their predecessors? Is the set of rationals really bigger than the set of counting numbers? Is the set of real numbers really bigger than the set of rationals? Is it possible that they are all simply instances of the mysterious "size" called infinity?

The man who first tackled these questions was Georg Cantor. Cantor was a German mathematician working in the second half of the 19^{th} century and the first two decades of the 20^{th} century. He was a contemporary of luminaries such as Poincaré, Kronecker, and Hilbert. The first two of these men refused to acknowledge his great contributions to mathematics; the third was an ally of

SECTION 3.5

RE-LEARNING TO COUNT
CONTINUED

tremendous standing. Cantor's work was controversial and his life was one of much struggle and little recognition. Denied employment at the more-respected universities in Germany, he was forced to work at smaller, less-prestigious institutions. Despite this, he helped set mathematics on firmer footing by fully examining the implications of using actually infinite sets. His first breakthrough was to re-define the concept of a number.

If you were asked to define the number 3, you could very well say "1, 2, 3", or "the number of things in the set {a, b, c}." Both of these responses are instances of the number three—both of them enumerate sets having three members, but neither of them defines the concept without referring to either "3" or "number." In general, we should be wary of definitions that must reference themselves. A better line of reasoning is required in order to come up with a true definition of a particular number.

Imagine that you are a ballroom dance teacher. As you begin a lesson, you want to make sure that you have the same number of girls and boys, so that each will have a dance partner of the opposite sex. You could take the time to count the boys, and then count the girls, and then compare the two numbers. A faster way would be simply to pair them off, one boy with one girl, until everyone has a partner. If there is no one left over, you have demonstrated that there are the same number of boys as girls. In mathematician's terms, you have shown a one-to-one correspondence between the set of boys and the set of girls in your dance class.

Going back to our troublesome definition, the most that we can say about the number three is that it is the property shared by the sets {1, 2, 3} and {*a, b, c*}, and all other sets that can be put into one-to-one correspondence with these sets. Hence, any set that can be put into one-to-one correspondence with these sets also shares the property of "three-ness". This is what we really mean when we say "three." Three is the common property of the group of sets containing three members. This idea is called "cardinality," which is a synonym for "size." The set {*a,b,c*} is a representative set of the cardinal number 3.

This all sounds like a bunch of semantics, but it is necessary to think of numbers in this way to gain a firm hold on the concept of infinity. We can use the technique of setting up one-to-one correspondence to compare the sizes of different sets without having to "count" all the members of the sets. This is indeed handy when it comes to infinity.

SECTION 3.5

RE-LEARNING TO COUNT
CONTINUED

COUNTING TO INFINITY

- The rational numbers can be put into one-to-one correspondence with the counting (natural) numbers.
- The irrational numbers cannot be put into one-to-one correspondence with the natural numbers.
- A "countable" infinite set is one that can be put into one-to-one correspondence with the set of natural numbers; an "uncountable" infinite set is one that cannot.

To get a sense of the tools we'll need to answer tough questions about infinity, we can start with a relatively straightforward example. Note that only 4 of the first 16 whole numbers are squares:

1, 2, 3, **4**, 5, 6, 7, 8, **9**, 10, 11, 12, 13, 14, 15, **16**, ...

It would be tempting to use this as evidence that there isn't one-to-one correspondence between the sets, as there seem to be more whole numbers than square numbers. However, in the 16th and 17th centuries, Galileo, famous for his work in astronomy and physics, demonstrated that there are, in fact, the same number of whole numbers and square numbers. To do so, he pointed out that every whole number can be made into a square number, after which it is possible to line up the two sets of numbers as so:

Whole Numbers	1	2	3	4	5	6
Square Numbers	1	4	9	16	25	36

Galileo's simple exercise made it clear that the set of whole numbers can be put into one-to-one correspondence with the set of square numbers. According to our new definition of number, this means that there must be the same number of each. What Galileo did was essentially to put the square numbers in a list and use the natural numbers to count them. Does this strategy also work for other types of numbers?

Let's consider the rational numbers for a moment. Given any fraction, we can always find a smaller one by taking half of it. So, if we want to list all of the rational numbers in sequence from least to greatest, which one should be first? We could say that $\frac{1}{1000}$ is pretty small and could potentially be first, but $\frac{1}{2000}$ is smaller—should it be first? This line of thinking obviously will not get us very

SECTION 3.5

RE-LEARNING TO COUNT
CONTINUED

far, as we can easily generate smaller and smaller fractions. Perhaps, because rational numbers are all expressible in terms of two quantities, a "counter" and a "namer," a single, linear list is not sufficient for the task at hand. It might be useful to organize fractions not in a list but, rather, in a two-dimensional array.

To accomplish this, imagine putting all of the fractions that have a 1 as their denominator in the first column of a table. Then let's put all the fractions that have a denominator of 2 in the second column, all those with a denominator of 3 in the third column, and so on. We will generate a table like this:

1/1	1/2	1/3	1/4	...
2/1	2/2	2/3	2/4	...
3/1	3/2	3/3	3/4	...
4/1	4/2	4/3	4/4	...
...	...	...	...	...

We could list every positive rational number if we were to continue this grid. Does this mean that the rational numbers cannot be put into a list, as is possible with the square numbers? Cantor, remarkably, showed that it is indeed possible to put rationals into a list format. This concept is now known as Cantor's "first diagonal" argument.

To compose such a list, we can trace out a weaving path through the table above, skipping over fractions that really are the same as ones we've already encountered (such as $\frac{2}{2}, \frac{2}{4}, \frac{4}{4}, \frac{1,000}{3,000}$, etc.).

1/1	1/2	1/3	1/4	...
2/1	2/2	2/3	2/4	...
3/1	3/2	3/3	3/4	...
4/1	4/2	4/3	4/4	...
5/1	5/2	5/3	5/4	...
...	...	...	...	...

This strategy creates a list that looks like this:
1/1, 2/1, 1/2, 1/3, 2/2, 3/1, 4/1, 3/2, 2/3, 1/4,...

It should be clear that writing the rationals in this fashion will account for every possible rational for as long as we care to continue.

SECTION 3.5

RE-LEARNING TO COUNT
CONTINUED

What's the advantage of having this list? It's easy to start counting them. We can assign the number 1 to $\frac{1}{1}$, the number 2 to $\frac{1}{2}$, the number 3 to $\frac{2}{1}$, and so forth for as long as we like. We see that there is a one-to-one correspondence between the natural numbers and the rational numbers. As with the boys and girls in the dance class example, the one-to-one correspondence indicates that the two sets are the same size.

So, the answer to the question, "How many rationals are there?" is "an infinite number," but it is a "countable" infinity. That is, it would be possible, in theory, to list all the rationals and to number them using the natural numbers. Any set that can be put into one-to-one correspondence with the set of natural numbers is considered to be countably infinite. Note that we have not mentioned negative rational numbers yet. However, through the same strategy they too can be put into a list that can be shown to have one-to-one correspondence with the set of natural numbers.

It might seem that we can put anything into a list that can then be matched up with the set of natural numbers. How about the set of all real numbers?

Again, we can look for a one-to-one correspondence with the natural numbers. Suppose we could list all of the real numbers, rational and irrational, between 0 and 1. Such a list, expressing both rationals and irrationals in decimal form, might look like this:

0.36264934...
0.11192737...
0.33333333...
0.66736270...
0.98800034...
...

Even though we are limiting ourselves to looking only between 0 and 1, Zeno made it clear that this list would be infinite. Furthermore, because we have put the numbers in a list, there should be a "first", "second", etc., on up to the "nth" number. So, it would appear that this list, as we have imagined it, is in one-to-one correspondence with the natural numbers and is, therefore, countable.

SECTION 3.5

RE-LEARNING TO COUNT
CONTINUED

However, at this point, Cantor had a second epiphany. He asked, "What if we create a new decimal by the following method?": In the columnar list of real numbers in decimal form, consider an array of diagonal digits—that is, the first digit of the first decimal, the second digit of the second decimal, the third digit of the third decimal, and so on.

0.36264934...

0.11192737...

0.66736270...

0.98800034...

In each case, if the digit is anything other than 1, put a 1 in the corresponding place of the new decimal number. If the digit on the diagonal is 1, put a 2 in the corresponding decimal place of the new number. The new decimal number formed in this way is different from every other number on this list by at least one digit.

0.~~3~~(1)6264934...
0.1~~1~~(2)192737...
0.66~~7~~(1)36270...
0.988~~0~~(1)0034...

→ New decimal not on the list
0.1211...

How can that be? The list we imagined was supposed to be complete, but we can clearly create a number that was not on that list! This reasoning is similar to the reasoning we employed with Euclid's proof of the infinitude of primes in the unit on prime numbers; namely, we start with a list that is assumed to be complete and then show that it isn't actually complete.
We now call this line of reasoning Cantor's "second diagonal" argument—it involves constructing a new number by following a diagonal path through the digits of a "complete" set.

Following this process proves that the original list was incomplete, because we were able to construct a number not in the list. This new number cannot be paired up with any of the natural numbers, because each of them is already

SECTION 3.5

RE-LEARNING TO COUNT
CONTINUED

paired up with a number from the original list. One could argue that we should simply make room for the new number on our original list and re-assign the pairings, but this could be done ad infinitum, and at every step we could still create a new number not in the list.

The only alternative explanation is that there must in some sense be more real numbers than natural numbers! In other words, the reals are not countable. Such a set is considered to be uncountably infinite.

We now have two distinct types of infinity, countable and uncountable. If we consider something that we cannot count to be larger than something that we can count (which seems logical), then it makes sense to say that the uncountable type of infinity is larger than the countable type.

Cantor called the cardinality of all the sets that can be put into one-to-one correspondence with the counting numbers $\aleph_0$, or "Aleph Null." The cardinality of sets that cannot be put into one-to-one correspondence with the counting numbers, such as the set of real numbers, is referred to as c. The designations $\aleph_0$ and c are known as "transfinite" cardinalities. The cardinality c is also known as the "cardinality of the continuum," denoting that these sets are best thought of as a continuous, unbroken line, as opposed to a discretely enumerated line. Such a line is akin to the Greek idea of a magnitude: infinitely divisible, with no discrete points. Both cardinalities are complete, actual infinities, rather than potential infinities, but they are not equal in size to one another.

The idea that there are different types of infinity might seem strange, but Cantor pushed his exploration even further into the realm of novel ideas. To understand more completely what Cantor contributed, we should ask at least two more questions.

First, we said that uncountable infinities, such as those with a cardinality of c, are "bigger" than countable infinities, such as those with a cardinality of $\aleph_0$, but we didn't prove it. Which is bigger, $\aleph_0$ or c? What does it mean for one number, finite or not, to be larger than another?

SECTION 3.5

RE-LEARNING TO COUNT
CONTINUED

Second, we must wonder whether there are any more types of infinity. That is, are there any more transfinite cardinalities? If not, why are there only two types?

These two questions are actually related. In the next section, we will see how their resolution leads to even more bizarre conclusions about the nature of the infinite.

SECTION 3.6

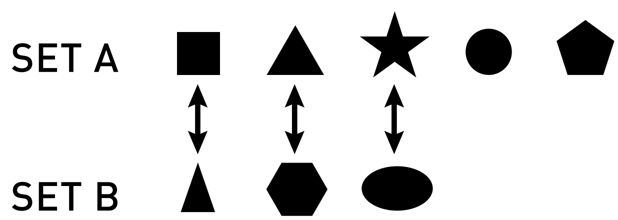
CANTOR'S THEOREM

- Which Is Bigger?
- Beyond Infinity

WHICH IS BIGGER?

- Set A is larger than set B if the elements of B can be put into one-to-one correspondence with a subset of the elements of A, but cannot be put into one-to-one correspondence with all of the elements of A.

What, in fact, is meant by the statement that one number is bigger than another number? In terms of cardinalities, this question basically means, "When is one set bigger than another?" Looking at sets of cardinality 5 and 3, for example, we see that:



For simplicity's sake, "3" means "a set of cardinality 3," and "5" means "a set of cardinality 5." So, "3" can be put into one-to-one correspondence with a subset of "5." Also, "3" cannot be put into one-to-one correspondence with all of "5." In other words, there is a one-to-one correspondence between "3" and part of "5," but not between "3" and all of "5." Intuitively, we can see that "5" must be bigger. This idea can be generalized. A set N is bigger than another set M, if:

There is a one-to-one correspondence between M and a subset of N.

AND

There is not a one-to-one correspondence between M and all of N.

We can use these conditions to consider the relationship between $\aleph_0$ and *c*. Let's choose a set of size $\aleph_0$, such as the natural numbers. The real numbers will be our set of size *c*.

SECTION 3.6

CANTOR'S THEOREM
CONTINUED

Checking for compliance with the first condition given above, we ask if there is a one-to-one correspondence between the natural numbers and a subset of the reals. Well, the set of real numbers includes the set of natural numbers, so the answer is "yes."

To verify that the second condition is also met, that there is not a one-to-one correspondence between the natural numbers and the reals, we can simply appeal to the same diagonal argument we made two sections ago. Recall that the natural numbers were not numerous enough to be put into one-to-one correspondence with the reals between 0 and 1. So, we have established firmly now that *c* is indeed larger than $\aleph_0$.

This exercise answers the first question posed at the end of the last section, and in doing so, it gives us a way to determine whether a certain set is bigger than another set. Let's now turn to the second question: could there be a transfinite cardinality larger than *c*? In other words, is there an infinity bigger than that exhibited by the set of real numbers?

BEYOND INFINITY

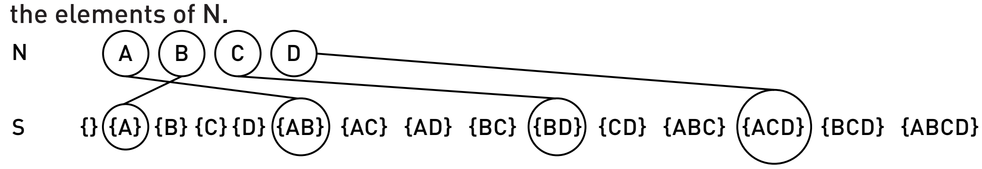
- Given a set of any non-zero size, it is possible to create a larger set by taking the set of subsets of the original.

Suppose we have a set N, consisting of *{A,B,C,D}*. We can then identify set S as the set of all subsets of N. We can conduct a little mental experiment by asking a friend to write the members of N in a row and then the members of S in another row below the first, like so:

N   A B C D

S   {} {A} {B} {C} {D} {AB} {AC} {AD} {BC} {BD} {CD} {ABC} {ABD} {ACD} {BCD} {ABCD}

Next, we ask the friend to circle four elements of S and to match them up with the elements of N.



SECTION 3.6

CANTOR'S THEOREM
CONTINUED

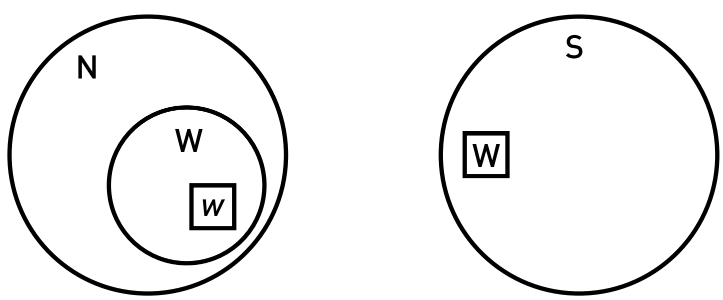
We can find an element of S that is not matched up by asking our friend whether an element of set N is matched with a subset that contains it. For example, say that our friend tells us "yes" for A, "no" for B, "no" for C, and "yes" for D. With this information, we can be sure that the subset {BC} has not been matched with any member of N. Because every element of N has been matched with an element of S, and there is at least one "leftover" element of S, we can say that S is definitely larger than N. Note that we did not have to count the members of either set to figure this out.

This strategy works for a finite set, but does it also work for an infinite set? The fact that we did not have to count anything to prove that S is bigger than N is a good sign. Let's see if anything changes if we let N be an infinite set and S be the set of subsets of N.

N={A,B,C,D, ...}
S={ {}, {A}, {AB}, {ABC}, ...}

Let's again attempt to match up every member of N with a member of S. We can ask the "yes or no" questions from before to find out which members of N are paired up with subsets that contain them. Let's say that any members of N for which the answer is "no" go into a new set called W. The set *W* is then the subset of N containing members that are not paired up with a subset that contains them.

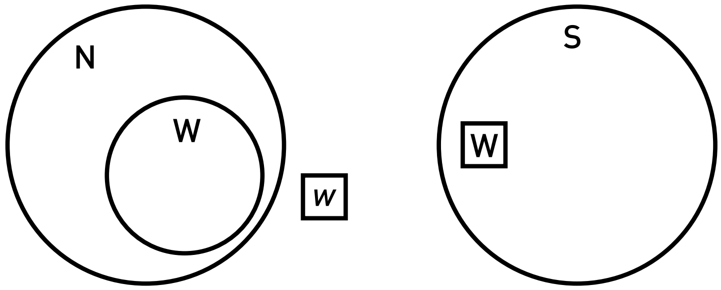
Because W is a subset of N, it must be in S, which is the set of all subsets of N. Can W be matched up with some element of N? Let's say that w, a member of N, is matched up with W.



If *w* is in W, then it is matched up with a subset that contains it, which by definition means that it cannot be in W; therefore, we have a logical contradiction.

SECTION 3.6

CANTOR'S THEOREM
CONTINUED



If, on the other hand, w is not in W, then it is not matched up with a subset that contains it, which is the requirement for being a member of W. However, we've already established that w is not in W, so again we have a logical contradiction.

Our only viable explanation is to assume that w cannot be matched up with W. This means that there are members of S that cannot be matched with any members of N; consequently, S must be bigger than N. We have now seen that in both the finite and infinite cases, the set of subsets is always larger than the original set.

To return to the cardinalities of infinity, we can create a set larger than $\aleph_0$ or c by taking the set of the subsets. We can clearly keep doing this as long as we please, so we have to conclude that there truly is an infinite number of different-sized infinities! This mind-boggling thought is one of the pure beauties of mathematics. It would be as if, having climbed the highest mountain in the world, one could see that there were peaks of heights previously unimaginable.

With his observations, Cantor spelled out the logical consequences of believing in actual infinity. Such an idea is "beyond human," which is partly why Cantor's ideas received so much criticism. In mathematics, however, there are many occasions to believe in concepts that are "beyond human," such as infinitely long lines in geometry (can anyone check?) or our modern acceptance of infinitely long decimals as numbers. Cantor showed that one can make logical sense out of infinity by thinking in terms of the sizes, or cardinalities, of sets. In doing this, he shored up the foundations of all the mathematical concepts that relied upon infinity. While there is still debate about some of the philosophical underpinnings of infinity in mathematics, most mathematicians do not have to concern themselves with it. This is due, in large part, to the work of Cantor. David Hilbert, one of the truly great mathematicians of Cantor's time, recognized the enormity of his contribution in the immortal thought:
"No one shall expel us from the paradise which Cantor has created."

SECTION 3.2

RATIONAL NUMBERS

- Rational numbers arise from the attempt to measure all quantities with a common unit of measure.
- Rational numbers can be expressed as decimals that repeat to infinity.
- In the mathematics of early Greece, there was a strong distinction between discrete and continuous measurement.
- Number refers to a discrete collection of atom-like units.
- Magnitude refers to something that is continuous and that can be infinitely subdivided.

SECTION 3.3

INCOMMENSURABILITY AND IRRATIONALITY

- The side and the diagonal of a square are incommensurable.
- Incommensurable quantities are not rationally related, because this logically leads to an infinite regress.

SECTION 3.4

ZENO'S PARADOXES

- The first of Zeno's arguments shows that considering a line segment to be a collection of points is contradictory.
- Zeno's paradoxes of motion are perhaps his most famous and extend his arguments to consider the absurdities of both discrete and continuous space and time.
- There are multiple ways to resolve Zeno's paradoxes, although many have shortcomings.
- The standard mathematical resolution uses the idea that a sum of infinite, decreasing, quantities can be considered finite.
- The idea of a limiting value to an infinite process is at the heart of calculus.

SECTION 3.5

RE-LEARNING TO COUNT

- To understand infinity, we need a new way to think about what a number is.
- The rational numbers can be put into one-to-one correspondence with the counting (natural) numbers.
- The irrational numbers cannot be put into one-to-one correspondence with the natural numbers.
- A "countable" infinite set is one that can be put into one-to-one correspondence with the set of natural numbers; an "uncountable" infinite set is one that cannot.

SECTION 3.6

CANTOR'S THEOREM

- Set A is larger than set B if the elements of A can be put into one-to-one correspondence with a subset of the elements of B, but cannot be put into one-to-one correspondence with all of the elements of B.
- Given a set of any non-zero size, it is possible to create a larger set by taking the set of subsets of the original.

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