MATHematics illuminated

PARTICIPANT GUIDE

UNIT 3

UNIT 03

HOW BIG IS INFINITY?

PARTICIPANT GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

ACTIVITY 1

We saw in the video and text that $\sqrt{2}$ is irrational. In text section 3.3, "Odds and Evens," we saw how to prove it by first assuming that $\sqrt{2} = \frac{a}{b}$ and then looking for a contradiction. Can we apply this method, or another similar method of proof, to show the irrationality of other numbers?

1. With your group, prove that $\sqrt[3]{21}$ is irrational.

2. Show that the thirty-first root of 15 must be irrational.

3. Say that a completely reduced fraction $\frac{a}{b}$ is equal to a whole number. Show that b must be 1.

4. Use your result from problem 3 to show that the following numbers are irrational:

$\sqrt{10}$

$\sqrt[3]{29}$

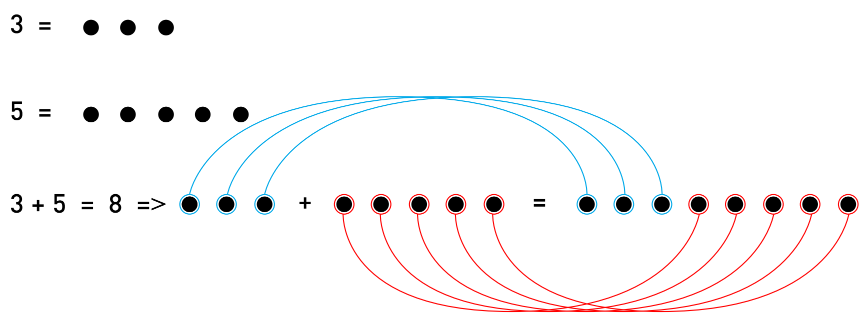
$\sqrt[5]{33}$

ACTIVITY 2

THE ARITHMETIC OF INFINITY

A

Let's represent the number "3" as three dots, the number "5" as five dots, and so on.

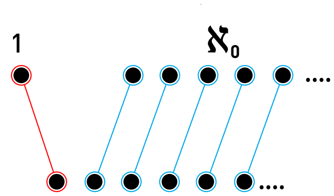


As shown above, we can also pictorially represent the arithmetical statement "3 + 5 = 8."

In the text you read about "countably infinite" sets. Georg Cantor used the symbol $\aleph_0$ to represent the cardinality, or size, of a countably infinite set. We can represent such a set as a string of dots that extends forever in one direction:

Let's explore arithmetic with $\aleph_0$.

The following diagrams show that $1 + \aleph_0 = \aleph_0$ and $\aleph_0 + 1 = \aleph_0$ (at least in terms of correspondences between representative diagrams of dots). Once a unit has been added to the infinite string of units, the result is indistinguishable from the original infinite sequence. In other words, an infinitely long string of beads is an infinitely long string of beads..



ACTIVITY 2

THE ARITHMETIC OF INFINITY
CONTINUED

1. Draw similar diagrams to show that $3 + \aleph_0$ and $\aleph_0 + 3$ each deserve to be called $\aleph_0$ (at least in terms of correspondences between representative diagrams of dots.)

2. Is there a way to interpret $\aleph_0 - 3$? Would it "equal" $\aleph_0$? Explain, using a picture as before.

3. Generalize what you found in the last question to make a statement about the sum or difference of $\aleph_0$ and a constant, b.

B

1. What picture represents $\aleph_0 + \aleph_0$? Show that $\aleph_0 + \aleph_0 = \aleph_0$.

2. Why does your answer to the above question imply that $2\aleph_0 = \aleph_0$?

3. Show that $3\aleph_0 = \aleph_0$.

4. Show that $\aleph_0^2 = \aleph_0$.

5. Show that $\aleph_0 - \aleph_0$ gives inconsistent answers.

6. Does it seem possible to give consistent meaning to "$\frac{\aleph_0}{\aleph_0}$"? Why or why not?

C

Discuss the classic "Hilbert's Hotel" problem.

A hotel has a countably infinite number of rooms numbered 1, 2, 3, ... One night, a traveler arrives late to find that the hotel is full.

1. Is there a way to accommodate the traveler? How?

2. A late train arrives at the hotel with 1,000 extra potential guests. Can they all be accommodated? How?

3. A train with a countably infinite number of people arrives. How can they all be accommodated?

ACTIVITY 2

THE ARITHMETIC OF INFINITY
CONTINUED

4. A countably infinite number of trains, each holding a countably infinite number of passengers, arrive. How can all the people be accommodated?

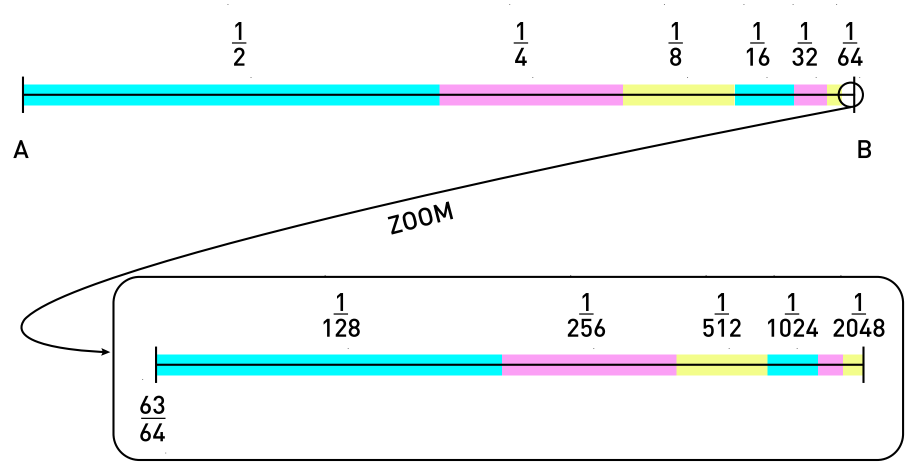
ACTIVITY 3

In Zeno's paradox The Dichotomy, a runner is trying to cover the distance from point A to point B but cannot get there because he must first cover half the distance. Of course, before he can get to the halfway point, he must first get to the quarter-way point (i.e., half of halfway). First, though, he must reach the eighth-way point, and so on for an infinite number of points.

A slightly different version of this paradox has the runner making it to the halfway point. To advance toward his ultimate goal, he then must cover half of the remaining distance ($\frac{1}{4}$ of the total distance), then half again ($\frac{1}{8}$), and so on ad infinitum, making it impossible to reach the final destination point. This version is closely related to the sum of a geometric series, an important concept that appears in multiple places in the textbook. In this activity, we will examine one of the ways in which this paradox is reputed to be resolved—that is, that the sum of an infinite number of steps can be a finite quantity.

A (5-10 minutes)

The fraction of the total distance that Zeno's runner must cover can be expressed as the series:



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

Let $S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$

ACTIVITY 3

1. Write an expression for 2S by doubling each term and showing that this new expression equals S+1. What then, must be the value of S? What does this say about Zeno's Dichotomy?

2. The above argument indicates only that the distance can be covered—however, it says nothing about time. How could you modify the above argument to show that this distance can be covered in a finite time?

3. Show that $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \frac{1}{2}$.

B

Let's come at the geometric series from a different angle. Imagine that you have a piece of paper and three friends. You are feeling especially generous and decide to share the paper equally with your friends.

1. You tear your piece of paper into four equal pieces and give one piece to each friend, keeping one for yourself. How much does each one of your friends have?

2. You then tear your remaining piece into four equal pieces and again distribute a piece to each friend, keeping one for yourself. Now how much does each one of your friends have?

3. How much do you have?

4. As time passes, what will happen to the paper?

5. Let's say that you and your friends are immortal and can continue to do this for all of time—what will happen to the paper?

6. Explain how this means that $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3}$.

7. Say instead that there are N people in your group of immortals. Show that if you are to play this "game," each person (other than you) will have $\frac{1}{(N-1)}$ of the paper. Explain any correlation between this and the general form of the resolution of Zeno's paradox.

8. What then does $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$ equal?

ACTIVITY 3

9. A general geometric series looks like:

$S_n = 1 + x + x^2 + \ldots + x + \ldots$

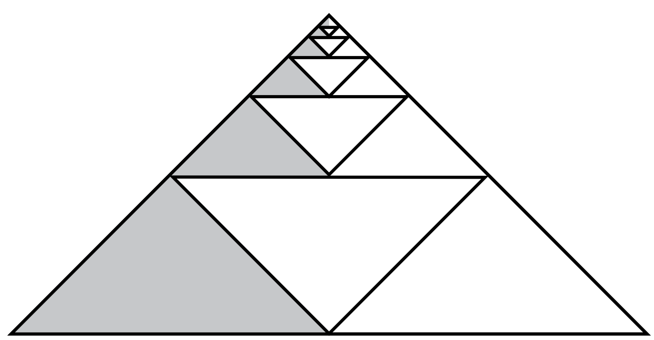
Extend the method you used in part A to show that $S_n = \dfrac{(x^{(n+1)} - 1)}{(x-1)}$.

Hint: Multiply both sides of the above equation by x to get a new equation. You now have two equations...

10. If $0 < x < 1$, the term x^{n+1} goes to zero as n grows larger and larger. What can we say about the "value" of $S = 1 + x + x^2 + x^3 + \ldots$ for this range of x-values? Put $x = \frac{1}{2}$ and $x = \frac{1}{3}$ and $x = \frac{1}{4}$ into this equation. What connections to the previous questions do you notice?

In your small group, discuss the relative merits of each of the three methods for determining the sum of a geometric series. Be prepared to share your conclusions with the large group.

C



1. Explain why the above figure represents the sum: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3}$

2. Come up with a similar figure to show: $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots = \frac{1}{2}$

ACTIVITY 4

How much space on the number line do the rational numbers take? How much space do the irrationals take?

In this activity we will explore some of the strange consequences of infinity by thinking about how countable and uncountable infinities behave on the number line. Try each problem and discuss your results and possible methods of attack in your small groups.

A

1. In your group show that between any two rational numbers lies another rational number.

2. Show that between any two points x and y on the real number line, there is a rational number, even if x and y are points representing irrational numbers.

Hint: Think about the decimal representations of x and y.

3. Generalize your findings from the questions so far.

4. Go further and show that between any two points, rational or irrational, there are an infinite number of rational numbers.

B

So, it seems that the rationals take up most of, if not almost all, the space on the number line. Let's see if this is really the case.

Note: For the sake of simplicity, we will focus on the positive half of the number line for the remainder of the discussion. The arguments work for the negative half also, of course, but they get a bit unwieldy.

Now that you have a list that contains every single rational number, let's imagine that there is a small "buffer zone" around each one's position on the number line.

1. Draw a number line and mark the positions of the first few rational numbers from your list.

ACTIVITY 4

2. On your number line, mark a "buffer zone" equal to $\frac{1}{10}$ of a unit around the point corresponding to 1.

3. Make a zone equal to $\frac{1}{100}$ of a unit around the second number on your list, a zone equal to $\frac{1}{1000}$ of a unit around the third, and so on...(just show the first four zones or so).

4. Now, as we theoretically draw more and more buffer zones, they will start to overlap, but we can find an upper limit to how much space they cover by adding them all together. Find this upper limit.

Hint: Start by writing the sum of all the buffer zones. Does this look familiar from any previous activities?

5. What does this mean about the total amount of space that the rational numbers occupy on the number line?

6. Use an argument similar to the one above to show that the rational numbers take up no more than $\frac{1}{99}$ of a unit on the number line.

7. If the rational numbers take up so little space on the number line, what occupies the rest?

CONCLUSION

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit, 3 How Big Is Infinity? could be related and brought into your classroom.

Questions to consider:

Which parts of this unit seem accessible to my students with no "frontloading?"

Which parts would be interesting, but might require some amount of preparation?

Which parts seem as if they would be overwhelming or intimidating to students?

How does the material in this unit compare to state or national standards? Are there any overlaps?

How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: Topology's Twists and Turns. Workshop participants are expected to read the accompanying text for Topology's Twists and Turns before the next session.

NOTES