MATHematics illuminated

PARTICIPANT GUIDE

UNIT 2

UNIT 02

COMBINATORICS COUNTS

PARTICIPANT GUIDE

ACTIVITIES

NOTE: At many points in the activities for Mathematics Illuminated, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

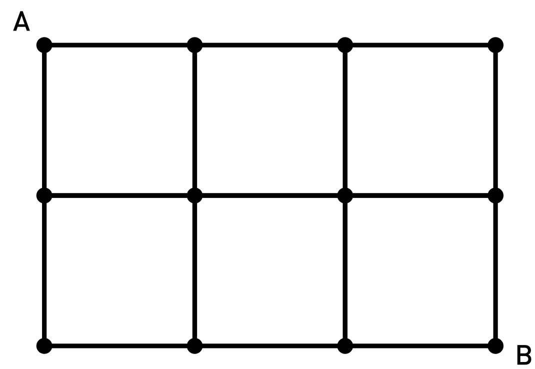
ACTIVITY 1

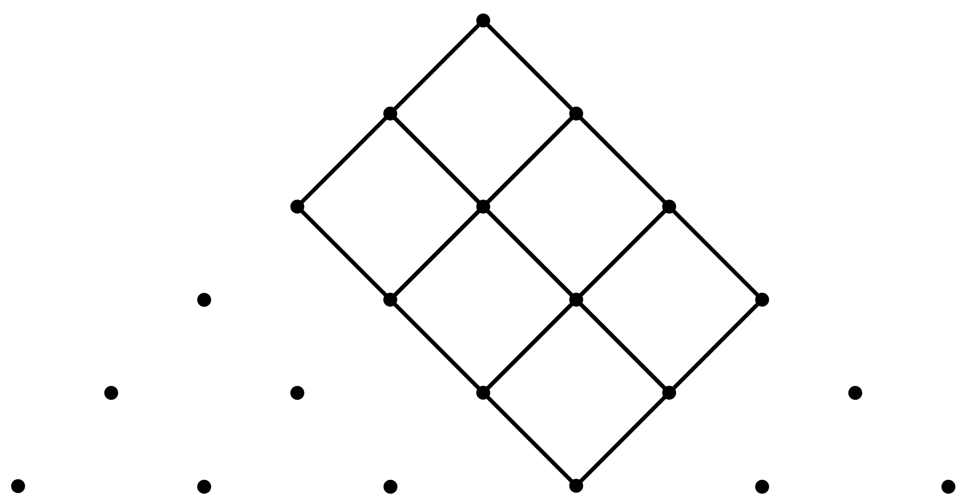
EXPLORING PASCAL'S TRIANGLE

In Combinatorics Counts you saw the connection between the counting function *C(n,k)*, the number of ways to choose a subset of k objects out of a set of n objects, and Pascal's Triangle, one of the most famous ideas in mathematics. Pascal's Triangle is a source of many mathematical mysteries, and in this activity you will explore some of the more surprising connections.

A

1. How many different shortest paths are there from point A to point B if you must stay on the grid lines?





On the above diagram, label the points of the triangle with numbers according to the pattern of Pascal's Triangle.

ACTIVITY 1

EXPLORING PASCAL'S TRIANGLE
CONTINUED

2. How does the number of shortest paths to point B relate to the number you found on the previous page? How do the numbers in Pascal's Triangle relate to the number of shortest paths between two points on a lattice?

3. Use the formula for *C(n,k)* found in the text along with the information you just acquired to write a general expression for the number of shortest paths from A to B on an M x N lattice.

B

Find the square and triangular numbers in Pascal's Triangle. Explain as precisely as possible.

C

Find the Fibonacci sequence (0, 1, 1, 2, 3, 5, 8, 13, 21, ...) in Pascal's Triangle. Explain.

Hint 1: recall that to find a number in the Fibonacci sequence, you just add together the two numbers that precede it. For example: 0 + 1 = **1**, 1 + 1 = **2**, 1 + 2 = **3**, 2 + 3 = **5**, etc.

ACTIVITY 1

EXPLORING PASCAL'S TRIANGLE
CONTINUED

D

Look at the prime-numbered rows (remember that the top of the triangle is considered to be row zero). What do you notice?

E

Compute the first 5 powers of 11, starting with the 0th power. What does this have to do with Pascal's Triangle?

ACTIVITY 2

Use the given K_6 graphs to play the game "Hexi."

THE HEXI GAME

MATERIALS

- Hexi game sheets
- Colored pencils, crayons, or markers

RULES

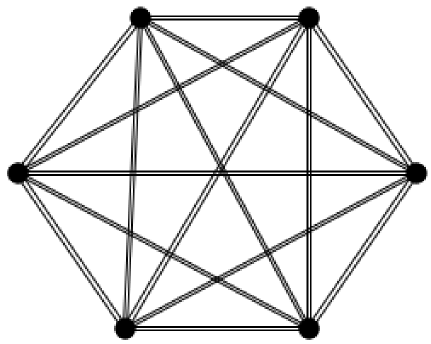
- This is a two-player game; each player should have a different color of pen, crayon, or marker.
- Decide who goes first.
- The first player colors one edge of the graph with her color.
- The next player colors a different edge of the graph with his color.
- The players alternate turns until all edges are colored. (Note: Players must color an edge when it is their turn.)
- The object is to force one's opponent to create a triangle of her own color. The first player to create a triangle of his own color loses.

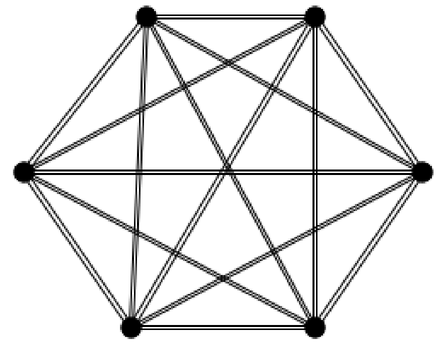
A Play this game with a partner from your small group. Play "best of three" before switching partners. Play until you figure out a winning strategy or until time runs out.

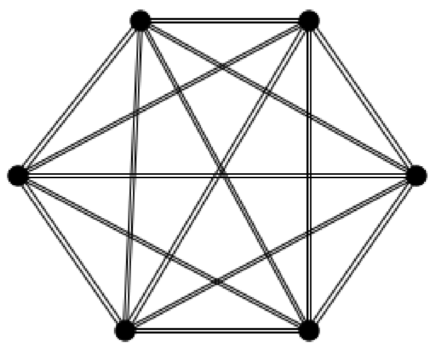
ACTIVITY 2

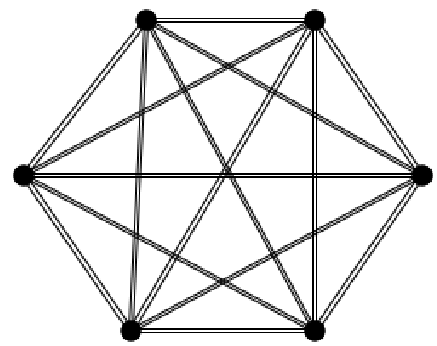
Hexi Game Sheet - Complete K_6 Graph

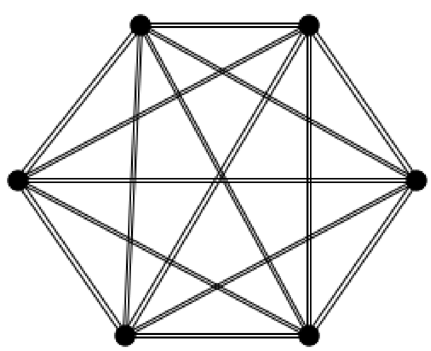
THE HEXI GAME
CONTINUED

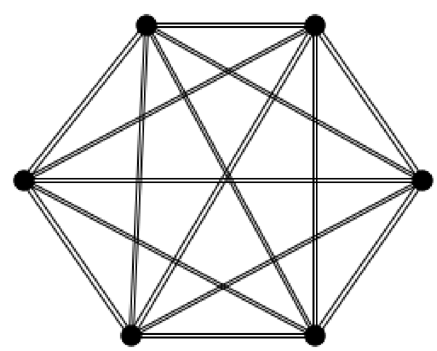












ACTIVITY 2

THE HEXI GAME
CONTINUED

In small groups, discuss the following questions:

1. What does this game have to do with the combinatorics unit?

2. Can there be a tie? Why not?

B

What is R(3,4)?

1. Draw K_3. How many possible colorings of K_3 are there using two colors?

2. Draw K_4. How many possible colorings of K_4 are there using two colors?

3. In words, what does R(3,4) represent?

4. Say that the two colors in R(3,4) are red and blue respectively. Can you find a coloring of K6 that has neither a red K_3 nor a blue K_4? Do the same for K_3, K_4, and K_5.

5. What have you just shown about the lower bound of R(3,4)?

Using the methods outlined in the textbook, show that K10 forces either a red K3 or a blue K_4.

6. First, look at the edges coming out of one vertex of K_{10}. If four of them are red, show that this guarantees either a red K_3 or a blue K_4.

7. After checking the possibility that four edges coming out of one vertex are red, what is the other scenario that we need to check?

8. Show that if six or more edges extending from one vertex are blue, a red K_3 or a blue K_4 is forced.

9. What does this say about the upper bound of R(3,4)?

10. What are the possible values for R(3,4)?

ACTIVITY 2

THE HEXI GAME
CONTINUED

11. How would you determine which of these is the correct value? What is your opinion of the difficulty of this task? (Hint: it might help to think about the number of possible colorings that have to be checked for counterexamples.)

12. Let's say that it turns out that R(3,4) is 9 (which it is). What is R(4,3)? Why?

ACTIVITY 3

FINDING DE BRUIJN SEQUENCES

Use the techniques from the text to find a sequence that contains all possible three-letter "words" that can be made with an alphabet consisting of just the letters A and B.

1. How many such three-letter "words" can be made?

2. What is the shortest length of a sequence that contains all possible three-letter words from above?

3. How many de Bruijn sequences are there (assuming an overlap of two)?

4. Describe your process.

5. Show that your sequence does indeed contain every three-letter "word."

6. If you take an Eulerian path instead of a Hamilton cycle, what does the resulting sequence represent?

Hint 1: When finding a de Bruijn sequence via the Hamilton cycle method, each vertex represents one of the "words." What does each edge represent?

Hint 2: How many edges are there on the directed graph? How many four-letter words can be made from the given alphabet?

ACTIVITY 4

MATERIALS

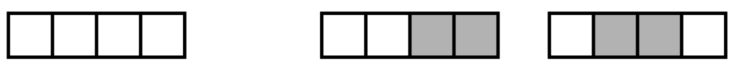
- Graph paper

One of the most famous sequences of numbers in mathematics is the Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, ... That this series appears in Pascal's Triangle, as you discovered in one of the mini-activities, is surely no coincidence. In this activity you will explore Fibonacci numbers from a combinatorial standpoint and then find out why there is a connection between the Fibonacci numbers and Pascal's Triangle.

A

1. How many ways can you express the number 4 as a sum of 1's and 2's?

A "4-board" is a string of four cells like so:



A 4-BOARD

2 POSSIBLE TILINGS OF A 4-BOARD

2. Show the sums you found in the first question as tilings of 4-boards. Note: we'll refer to single squares as just "squares" and double squares as "dominoes."

Let's call the number of tilings of a 4-board using squares and dominoes f_4.

3. Use 3-boards to find f_3.

4. Use 5-boards to find f_5.

5. Write an expression for f_5 using f_3 and f_4.

6. If f_5 is replaced with fn, what would the above expression become?

ACTIVITY 4

B

1. On an n-board, if the first tile is a square, how many ways are there to complete the board?

2. On an n-board, if the first tile is a domino, how many ways are there to complete the board?

3. Write an expression for the number of tilings of an n-board. (Hint: think of it as the sum of the n-boards with a square as the first tile and the n-boards with a domino as the first tile.

4. If we define f-1 = 0 and f0 = 1, start with f1 and write the first ten fn's.

5. Does this series look familiar?

C

We can use the fact that upward diagonal sums in Pascal's Triangle correspond to the Fibonacci numbers to write an expression for the nth Fibonacci number.

1. How many 12-tilings use two dominoes and eight squares? three dominoes and six squares? k dominoes and 12-2k squares?

2. How many n-tilings use one domino? two dominoes? k dominoes?

3. Write an expression for f_n in terms of the binomial coefficient, C(n, k).

Hint 1: Count the number of n-tilings by considering the number of dominoes used.

4. Compare and contrast the expressions for f_n found earlier in parts A and B.

ACTIVITY 4

IF TIME ALLOWS:

For n greater than or equal to zero, show that $f_0 + f_1 + f_2 + f_3 + \ldots + f_n = f_{n+2} - 1$.

Hint 1: In terms of n-boards, what does f_{n+2} represent?

Hint 2: How many (n+2)-boards use at least one domino? It might help to think about how many tilings use no dominoes.

Hint 3: Answer the question in Hint 2 another way: let the last domino fall in cells k+1 and k+2—all the squares after k+2, up to cell n, are then squares, but the cells before k+1 can be a mix of squares and dominoes. How many tilings are there? Perhaps start with the last domino in cells 1 and 2, then 2 and 3, then 3 and 4, etc.

CONCLUSION 4

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 2, Combinatorics Counts could be related and brought into your classroom.

Questions to consider:

1. Which parts of this unit seem accessible to my students with no "frontloading?"

2. Which parts would be interesting, but might require some amount of preparation?

3. Which parts seem as if they would be overwhelming or intimidating to students?

4. How does the material in this unit compare to state or national standards? Are there any overlaps?

5. How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS

Please use the last 30 minutes of class to watch the video for the next unit: How Big Is Infinity? Workshop participants are expected to read the accompanying text for How Big Is Infinity? before the next session.

NOTES