MATHematics illuminated

FACILITATOR GUIDE

UNIT 1

UNIT 01

THE PRIMES

FACILITATOR GUIDE

ACTIVITIES

NOTE: At many points in the activities for *Mathematics Illuminated*, workshop participants will be asked to explain, either verbally or in written form, the process they use to answer the questions posed in the activities. This serves two purposes: for the participant as a student, it helps to solidify any previously unfamiliar concepts that are addressed; for the participant as a teacher, it helps to develop the skill of teaching students "why," not just "how," when it comes to confronting mathematical challenges.

NOTE: Instructions, answers, and explanations that are meant for the facilitator only and not the participant are in grey boxes for easy identification.

ACTIVITY 1

HOW TO COMMUNICATE WITH ALIENS

(45 minutes)

MATERIALS

- Graph paper
- Pencil or pen

A (15 minutes)

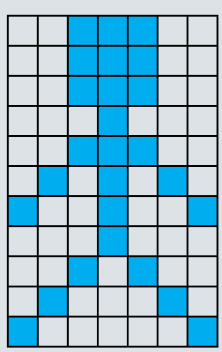
Imagine that you are a SETI (Search for Extra-Terrestrial Intelligence) researcher, and you receive the following sequence of information. You suspect that it might be a coded message, but you are unsure how to decipher it.

MESSAGE 1

0011100001110000111000000100000011100010101010010010001000000101000100
0101000001

Working in a small group, use the relationship between the Drake pictogram and prime factors—as discussed in section 1.1 of the text—to make sense of this alien communication.

Answer:



Item 3203 / HUB Collective LTD., created for *Mathematics Illuminated*, IMAGE 1.1 (2008).
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ACTIVITY 1

HOW TO COMMUNICATE WITH ALIENS
CONTINUED

Facilitator's note: encourage the participants to look for clues in the length of the code. Let them play with it for about 5 minutes before prompting any groups that haven't made progress.

Refer to each digit in the code as a "bit."

For groups who need some prompting, you might ask them whether the number of bits in the code is factorable. Then you could ask them if it is factorable in multiple ways. If it is factorable in only one way (which it is)—that is, if it has only prime factors—it might make sense to break it up into chunks and try to arrange the chunks in some systematic fashion. In the case of this example, the string is 77 bits long, which factors into 11 groups of 7 or 7 groups of 11.

Let the first seven bits correspond to the first row of a 7 x 11 rectangle on the graph paper. The second seven bits make up the second row, and so on. After arranging all of the bits in 11 rows, color each grid square that corresponds to a "1" in the code to create the hidden picture "message."

Convene the large group and discuss the method used to crack this code and the role that prime numbers play.

B (10 minutes)

In this exercise, you will use the de-coding method you used in part A to decipher a coded message.

Facilitator's note: assign one of the following coded messages to each group. Ideally, each group has a different code.

Cut out the five following code cards and hand one to each group:

ACTIVITY 1

HOW TO COMMUNICATE WITH ALIENS
CONTINUED

CODE 2

0000000000000000000000000000000000000000011111000000000000110000011000
00001000000000010000010000000000001000010011000110010001000110001 1000
1001000000000000001001000000000000001001000000000000001001001000000010
0100010011000110010000100001110000100000100000000010000000110000011
0000000000111110000000000000000000000000000000000000000000

CODE 3

000000000000000000111000000000110001100001100000001101000001000000110
0100000100101000000000100011110111110000000010000000000000100000000000
1111100000001011101000001001110010000000111000000000001010000000000001
01000000000001010000000001101100000000000000000

CODE 4

00000000001000000000000110000000000010100000000110001000000001100010000
00000000100000000000111110000000000000000000000001110000001101000100000 1
1000001000000000000100000011000100000000110010000000000001111100000000000
00000000000000111000000110100010011011000001001100000001100000001100000
1000001101000100000000001110

CODE 5

000000000000000000000000001101100000000000000100100100000000000000100000
00100000000000101101101000000000001001010010000000000001000001000000000
000010001000000000000000010001000000000000000001010000000000000000010000
0000000000000010000000000000000001111100000000000000101110100000000000001
0001000100000000000000111110000000000000010111010000000000000000010000000
00000000011111000000000000001011101000000000000000101000000000000000001
0100000000000000001101100000000

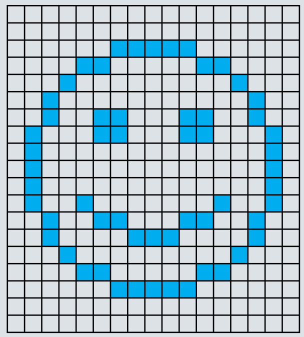
CODE 6

000000001000000000000000001010000000000000000101000000000000001000100000
000000010001000000000000100000100000000000100000100000000010000000100
000000101101101000000001001101100100000010000000001000001000000000000
10000111111111111100000000001000000000000000111110000000000000010111010
000000000001010000000000000000101000000000000001101100000

ACTIVITY 1

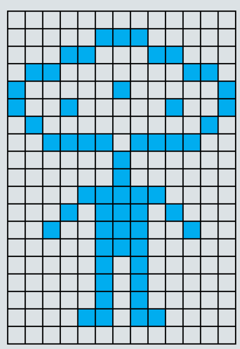
HOW TO COMMUNICATE WITH ALIENS
CONTINUED

Answer: Code 2



Item 3204 / HUB Collective LTD., created for *Mathematics Illuminated*, IMAGE 1.2 (2008).
Courtesy of Oregon Public Broadcasting.

Answer: Code 3



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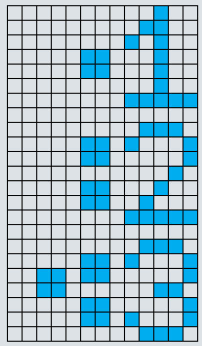
ACTIVITY 1

HOW TO COMMUNICATE WITH ALIENS
CONTINUED

Answer: Code 4



Item 3206 / HUB Collective LTD., created for *Mathematics Illuminated*, IMAGE 1.4 (2008).
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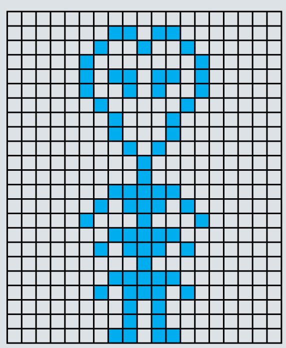
Answer: Code 5



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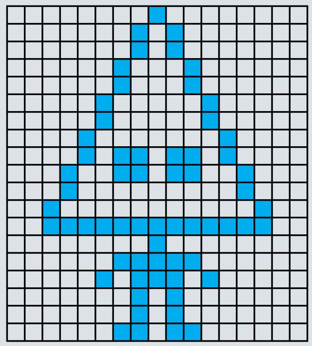
ACTIVITY 1

HOW TO COMMUNICATE WITH ALIENS
CONTINUED

Answer: Code 6



Item 3202 / HUB Collective LTD., created for *Mathematics Illuminated*, IMAGE 1.6 (2008).
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C (15 minutes)

In your group, design a coded message to send to another group. Your code should be a string of ones and zeros, with fewer than 350 bits. In designing your message and code, be sure to use what you know about prime factors.

When you are finished, find another group who is finished and exchange codes. See if you can decipher the code to recreate the other group's original pictogram message.

D (5 minutes)

Discussion: Would alien intelligence really be able to interpret these "prime factor pictograms?" If we assume that they can, what assumptions are we making about the aliens?

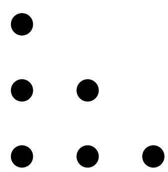
These questions may have many answers. To prompt the group, think about whether or not a civilization needs math for technology and whether or not a prime number would be recognized as "special" by an alien intelligence.

ACTIVITY 2

(30 minutes)

One of the themes of Unit 1: "The Primes" is the study of numbers for their own sake. In this activity, we will explore the relationship between two famous types of "figurate" numbers.

Recall from the text that a triangular number is a number that, when represented by a collection of dots, can be arranged in the shape of a triangle. An example of a triangular number is the number six, as shown in this dot pattern:



1. List the first ten triangular numbers.
Answer: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

2. Given the nth triangular number, how can you find the next one?
Answer: Add n+1 to the nth triangular number. Example: six is the third triangular number; to find the fourth, add four to six to get ten.

3. List the first ten square numbers.
Answer: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

4. Sum together any two consecutive triangular numbers. What do you notice? Try it again. Can you explain what you have observed?
Answer: Answers will vary.

5. Use the relationship between square and triangular numbers that you found in question 4 to write a formula for the nth triangular number in terms of n only.

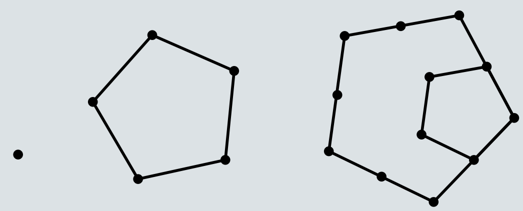
Answer: $\frac{n(n+1)}{2}$. The participant can find this by looking at square numbers and dividing them in "half" diagonally (sort of). First, notice that the (n+1)th square number can be broken into the sum of the nth triangular number and the (n+1)th triangular number. Express the (n+1)th square number as a sum of two triangular numbers and solve for the nth triangular number.

Convene the group and discuss.

ACTIVITY 2

6. Give some examples of pentagonal numbers.

Answers: 1, 5, 12, 22, 35, ...



Pentagonal numbers

Item 3214 / HUB Collective LTD., created for *Mathematics Illuminated*, IMAGE 1.7 (2008).
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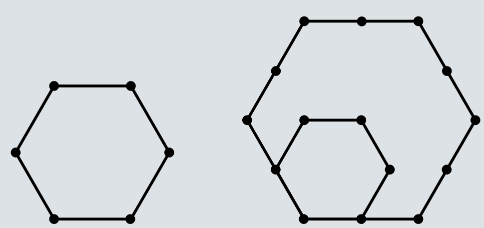
7. Give some examples of hexagonal numbers.

Answers: 1, 6, 15, 28, 45, ...



Hexagonal numbers

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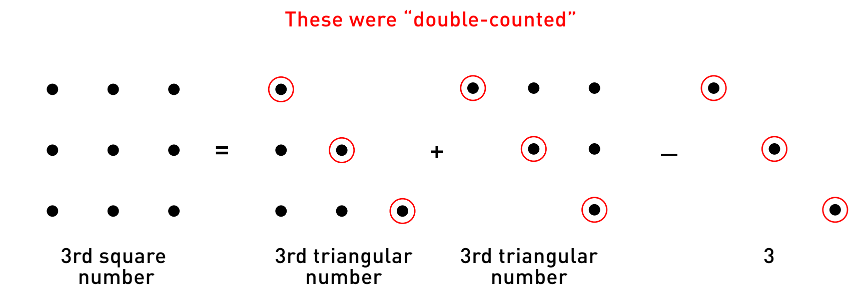
ACTIVITY 2

Notice that s_n, the n^{th} square number, is equal to the sum of two of the n^{th} triangular numbers minus the diagonal (because it is double-counted).



Generalizing the above diagram, we can say that $s_n = 2\ t_n - n$.

8. Find similar formulas for the n^{th} pentagonal and n^{th} hexagonal numbers.

Hint 1: For the pentagonal numbers, start with three of the n^{th} triangular numbers. For the hexagonal numbers, start with four of the n^{th} triangular numbers.

Answer: Each pentagonal number is $p_n = 3t_n - 2n$ (three triangles with a double count of two "seams" of n dots). Each hexagonal number is $h_n = 4t_n - 3n$.

9. Write a similar formula for the nth m-gonal number. (an m-gon is a regular polygon with m sides.)

Answer: $m_n = (m\text{-}2)t_n - (m\text{-}3)n$

ACTIVITY 3

(20 minutes)

MATERIALS

- A deck of playing cards for each group (or, as an alternative, about 30 pennies for each group)

Facilitator's note:
This exercise works best with groups of no more than three individuals.

This activity is a demonstration of the famous "Locker Problem" and can be used to help explore the factor-structure of the natural numbers.

A (10 minutes)

The facilitator will hand each group a deck of playing cards. Take 25 of the cards and lay them face up in one row. It doesn't matter which cards you use.

Use any method you like to decide who goes first.

The first person should begin at the left end of the row of cards and work his or her way down the row, turning every card face down.

The second person should then start at the left end and turn over every second card. (Every second card should be face up at this point.)

The third person should go next, again starting at the left end of the row, this time turning over every third card. Note that this will require turning some cards face up and other cards face down—let's call this act of turning a card over "changing the card's state."

Continue taking turns changing the state of specific cards: on the fourth turn, the person should change the state of every fourth card; on the fifth turn, every fifth card; and so on up until the twenty-fifth turn.

1. What are the positions of the cards that are left face down?
Answer: The positions of the cards left face down correspond to the square numbers: 1, 4, 9, 16, 25

ACTIVITY 3

2. Explain.
Answer: Answers will vary, but here's an example:
Square numbers have an odd number of unique divisors. A card's state changes once for every unique divisor of its position number.

Discuss the various answers and methods as a large group. In the discussion, the methods are as important as the answers.

3. Does it matter which order the participants go in when flipping? For example, flip every third card first, then every fifth, then every second, and so on for all numbers up to 25. Try it! Explain what you find and why.
Answer: The order does not matter, as long as they all go.

B (10 minutes)

Try this activity one more time, this time keeping track of how many times each card is flipped. (You might want to appoint a group "secretary" to keep a tally sheet for each position number.)

1. Which card was flipped exactly once?
Answer: The first card.

2. Which cards were flipped exactly twice?
Answer: The cards in prime positions: 2nd, 3rd, 5th, 7th, ...23rd

3. Suppose we throw away all the cards that were flipped three or more times. What do the remaining cards have in common?
Answer: They are all in prime positions.

Facilitator's note: convene the class and discuss how this activity relates to the famous "Sieve of Eratosthenes," which was mentioned in the video and textbook for this unit. This is a method for finding prime numbers in which one starts with a list of all natural numbers and then eliminates first all the multiples of two greater than two—then all the multiples of three greater than three—then all the multiples of four greater than four—etc. The numbers left are the prime numbers.

ACTIVITY 4

FINDING PRIMES

(20 minutes)

MATERIALS

- Scientific calculator
- Graph paper

One aspect of prime numbers that is of interest to mathematicians is how they are distributed on the number line. If there is a pattern behind the distribution of the primes, it has eluded the greatest minds in mathematics for thousands of years. We have nice formulas that can tell us the nth square number, or the nth triangular number, but what about the nth prime number?

One way to approach this problem is to think about the number of primes below a certain number, N. Let's say that there is a function $\pi(N)$ that gives the number of primes below N.

1. Use the following list of primes to make a graph of N vs. $\pi(N)$ for N = 0 --> 200 with N as the horizontal. Choose reasonable increments for the horizontal and vertical axes.

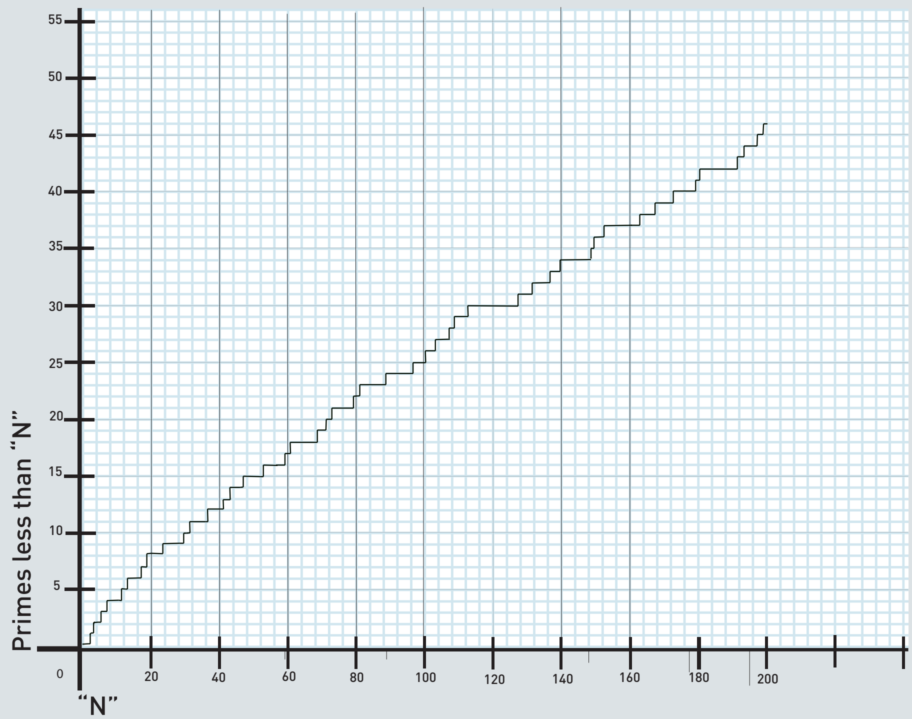
List of prime numbers less than 1,000:

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997		

ACTIVITY 4

FINDING PRIMES
CONTINUED

Answer: Graph of Prime Density



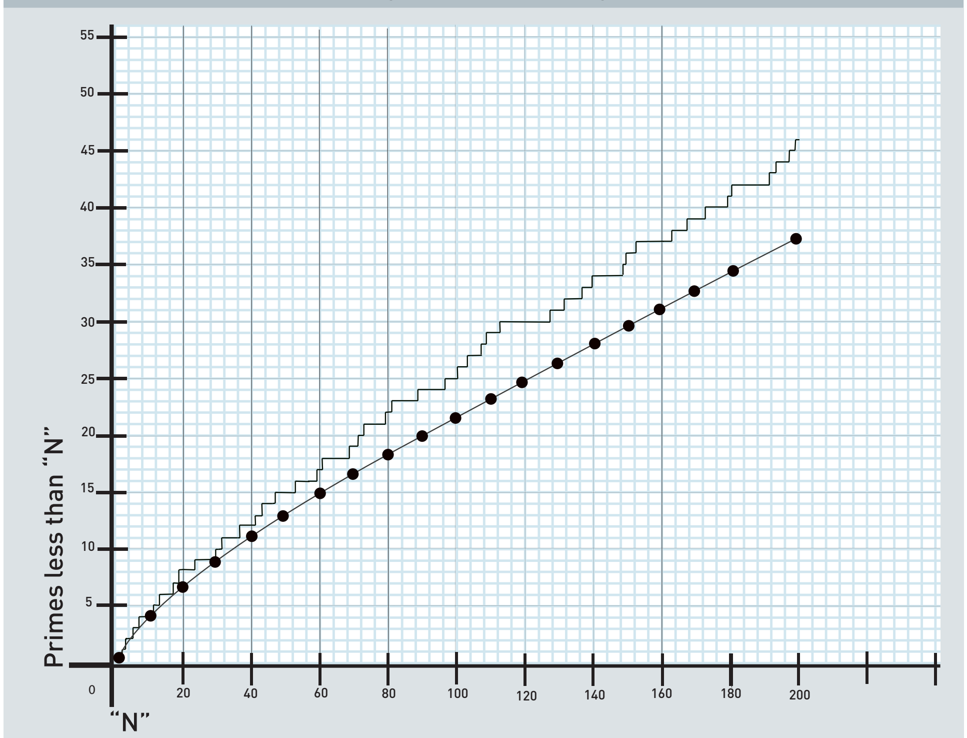
Item 2564 / Oregon Public Broadcasting, created for *Mathematics Illuminated*, 1.9 (2008).
Courtesy of Oregon Public Broadcasting.

ACTIVITY 4

FINDING PRIMES
CONTINUED

2. The graph you made in question 1 looks somewhat like a staircase; it would be very difficult to model it accurately with a simple function. The great Karl Gauss approximated it with $\pi(N) \sim N/(\ln N)$ by noticing that the distribution looks somewhat logarithmic. On the same graph that you made in question 1, plot 20 values of Gauss's approximation, using values of N equally spaced between 0 and 200. How good of an approximation is this? Describe what happens to the approximation as N gets larger.

Answer: Gauss's approximation gets better for larger values of N.



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ACTIVITY 4

FINDING PRIMES
CONTINUED

3. By examining how the ratio of N/ π(N) changes as N increases, one can derive a relatively famous result for a function that will give an approximation of the Nth prime:

N*th* prime ~ N x ln N

Choose about ten values from the given list of primes and compare them to this approximation. Be sure to choose a wide range of values. As N gets larger, describe what happens to the approximation for the Nth prime.
Answer: Answers will vary, but be sure that participants use a number's position, not its value, in computing the approximation. Example: 7 is the fourth prime, so the approximation is 4 x ln(4) ~ 5.55, which is off by about 21%. 19 is the eighth prime, and its approximation, about 16.7, is off by about 12%. In terms of percentage, the approximation gets better for larger values of N.

CONCLUSION (30 minutes)

DISCUSSION

HOW TO RELATE TOPICS IN THIS UNIT TO STATE OR NATIONAL STANDARDS

Facilitator's note:
Have copies of national, state, or district mathematics standards available.

Mathematics Illuminated gives an overview of what students can expect when they leave the study of secondary mathematics and continue on into college. While the specific topics may not be applicable to state or national standards as a whole, there are many connections that can be made to the ideas that your students wrestle with in both middle school and high school math. For example, in Unit 12, In Sync, the relationship between slope and calculus is discussed.

Please take some time with your group to brainstorm how ideas from Unit 1, The Primes could be related and brought into your classroom.

Questions to consider:

1. Which parts of this unit seem accessible to my students with no "frontloading?"

2. Which parts would be interesting, but might require some amount of preparation?
3. Which parts seem as if they would be overwhelming or intimidating to students?

4. How does the material in this unit compare to state or national standards? Are there any overlaps?

5. How might certain ideas from this unit be modified to be relevant to your curriculum?

WATCH VIDEO FOR NEXT CLASS (30 minutes)

Please use the last 30 minutes of class to watch the video for the next unit: Combinatorics Counts. Workshop participants are expected to read the accompanying text for Combinatorics Counts before the next session.

NOTES