Consider first the intuitive approach: Think hard about a lot of different cases and decide what is the right way to add and multiply in each one. To use intuition, you need to think in terms of some concrete interpretation of arithmetic. The yield of financial transactions is a good one for these purposes. Here negative amounts are money you owe, and positive amounts are money that you have or are owed by someone else. If you owe \$2 to Joan, and \$3 to Sammy, then you owe \$5 to the two of them together. So (-2) + (-3) = -5. If you owe \$2 to three people, then you owe \$6, so 3 × (-2) = -6. If you have a debt of \$2 and someone takes it away, you have gained \$2. So –(-2) = 2. If someone takes three \$2 debts away from you, the amount you owe is then \$6 less than before, which means you have \$6 more. Therefore (-3) × (-2) = 6. Continuing in this way, you can puzzle out what the sum, difference, or product of any two integers should be. The trouble with this approach is that it is somewhat contrived and depends upon making decisions about how to interpret each case in the particular context.[5]

Another approach[6] is to use an exploratory method to reason how the operations should extend from the whole numbers. By extending the patterns in the table below, you find that (-3) × (-2) = 6, just as was shown above in context.

3 + 2 = 5	3 – 2 = 1	3 × 2 = 6	(-3) × 2 = -6
3 + 1 = 4	3 – 1 = 2	3 × 1 = 3	(-3) × 1 = -3
3 + 0 = 3	3 – 0 = 3	3 × 0 = 0	(-3) × 0 = 0
3 + (-1) =	3 – (-1) =	3 × (-1) =	(-3) × (-2) =
3 + (-2) =	3 – (-2) =	3 × (-2) =	(-3) × (-2) =

By means of somewhat lengthy reasoning, you can find out how to do arithmetic with integers. But are the regularities observed about the whole number system (the rules in Box 3-1) still valid? Going through the cases again will show that they are. So not only has the number system been extended from the natural numbers to all integers, but the arithmetic in the larger system looks very similar to arithmetic in the original one in the sense that these laws are still valid.

Moreover, there are some new notable regularities that describe how the new numbers are related to the original ones. These are summarized in Boxes 3-2 and 3-3.

Box 3-2

Additional Properties of Addition

Additive identity. Adding zero to any number gives that number. For example, $3 + 0 = 3$ and $0 + 3 = 3$. In general, $m + 0 = m$, and $0 + m = m$.

Additive inverse. Every number has an additive inverse, also called an opposite. The opposite is the unique number that, when added to that number, gives zero. For example, the opposite of 3 is -3 because $3 + -3 = 0$; the opposite of -4 is 4 because $-4 + 4 = 0$. In general, $-s$ is the unique solution m for $s + m = 0$.

Box 3-3

Consequences of the Basic Properties: Formulas for the Arithmetic of Negation

Subtraction and negation. Subtracting a number is the same as adding its opposite. For example, $5 - 3 = 5 + (-3)$ and $5 - (-2) = 5 + 2$. In general, $s - t = s + (-t)$.

Multiplication and negation. Negation is the same as multiplication by –1. For example, $-3 = (-1) \times 3$ and $2 = (-1) \times (-2)$. In general, $-s = (-1) \times s$.

Opposite of opposite. The opposite of the opposite of a number is the number itself. For example, $-(-3) = 3$. In general, $-(-s) = s$.

Something much more dramatic is also true. One can show that, if the goal is to extend addition and multiplication from the whole numbers to the integers in such a way that the laws of arithmetic of Boxes 3-1 and 3-2 remain true, then *there is only one way to do it*. And the rules in Box 3-3 describe how it has to work. Recipes laboriously constructed by means of some sort of concrete interpretation of negative numbers are all completely dictated by this short list of rules of arithmetic. This uniqueness is a striking exhibition of the power of these rules—that they capture in a few general statements a large chunk of people's intuition about arithmetic. The extension of whole numbers to integers is an example of the axiomatic method in mathematics: basing a mathematical system on a short list of key properties. Its most famous success is the *Elements* of Euclid for plane geometry. Since Euclid's time, axiomatic schemes have been constructed to cover most areas of mathematics.

Another rather striking thing has happened during this extension from whole numbers to (all) integers. The reason for making the extension was to be able to solve subtraction problems. Now, in the integers, subtraction is a true operation in the sense that you can subtract any integer from any other. As described in the rule on additive inverses in Box 3-2, for every integer, there is another integer, called its *opposite* or *additive inverse*, that counterbalances it: the two sum to zero. Thus 2 + (-2) = 0, and -84 + 84 = 0. The second equation means that –(-84) = 84 and leads to the rule on subtraction and negation in Box 3-3, which says that *subtracting an integer gives the same result as adding its additive inverse*. Thus 2 – 3 = 2 + (-3), and 24 – (-7) = 24 + (-(-7)), which is equal to 24 + 7 = 31. Thus, at least on a conceptual level, subtraction is merged into addition, and you really only need to have the single operation of addition to capture all the arithmetic of addition and subtraction. As soon as subtraction is made into a true operation by extending the whole numbers to the integers, you also get additive inverses, which allows you to subordinate subtraction to addition. This sort of simplification illustrates a kind of mathematical elegance: Two ideas that seemed different can be subsumed under one bigger idea. As we show below, the analogous thing happens to division when you construct rational numbers. That subordination is the best justification for why mathematicians talk about only the two operations of addition and multiplication when discussing number systems, and not all four operations recognized in school arithmetic.

Division and Fractions

Forgetting for a moment the triumph with integers, return to the whole numbers and the problem of division. Here the situation is in some sense much more complicated than for subtraction. You can subtract in whole numbers about half the time. However, division of one whole number by another rarely comes out even. If I have eight apples and want to share them equally with Carl and Maria (the three of us), I either have to leave two apples out of the division or have to cut them in pieces. The desire to solve this kind of problem leads to new numbers, the *positive rational numbers*. These are usually written as fractions (here we allow improper fractions, such as 12/5, in which the numerator is larger than the denominator), and each one is a solution to a division problem for integers. For example, $\frac{2}{3}$ is the number you get when you divide 2 into 3 equal parts. In other words, $\frac{2}{3}$ is by definition the number such that $3 \times \frac{2}{3} = 2$. Although this definition suffices to specify fractions as mathematical objects, fractions have many concrete interpretations. We refer the reader to the section "Discontinuities in Proficiency" in chapter 7 for a list of such interpretations.

Again, having introduced these new numbers, you find yourself needing to do arithmetic with them. If I get half an apple from Bart and two thirds of an apple from Teresa, how many apples do I have? If I have $1\frac{3}{4}$ boxes of marbles, and I want to put them in boxes half as large, how many of the small boxes will that make? By figuring out the answers to these questions, you turn the positive rational numbers (along with zero) into a number system, with operations of addition and multiplication extending the old operations on whole numbers. This feat is difficult technically and conceptually. The arithmetic of, and even developing meanings for, fractions is one of the stumbling blocks of the pre-K to grade 8 mathematics curriculum.[7]

Nevertheless, if you go through the effort of constructing the arithmetic of positive rational numbers by considering various cases and using some sort of concrete model, as with the integers, you find that it can be done. At the end of your labors, being a mathematician, you survey the new system and ask whether the marvelous rules of Box 3-1 still hold. They do! Moreover, there are some further regularities, analogous to the rules of Box 3-2, that relate the new numbers to the old. The new rules for multiplication are listed in Box 3-4.

Box 3-4

Additional Properties of Multiplication

Multiplicative identity. Multiplying a number by 1 gives that number: $5 \times 1 = 5$ and $1 \times 5 = 5$. In general, $m \times 1 = m$ and $1 \times m = m$.

Multiplicative inverse. Every number other than 0 has a multiplicative inverse, also called a reciprocal. The reciprocal is the unique number that, when multiplied by that number, gives 1. For example, the reciprocal of 3 is 1/3 because $3 \times \frac{1}{3} = 1$; the reciprocal of 5/8 is 8/5 because $\frac{5}{8} \times \frac{8}{5} = 1$. In general, for s not zero, $1/s$ is the unique solution m of $s \times m = 1$.

The analogy with the construction of the integers is remarkable, with multiplication replacing addition, and division replacing subtraction. First, the arithmetic in the laboriously constructed new system is entirely determined by the rules of Boxes 3-1 to 3-4. This means that for the formulas of adding, multiplying and dividing (positive) rational numbers, as described in Box 3-5, there really was no choice: That is the only way to do it and preserve the rules.[8] Furthermore, although the new system was created to allow division, once you have it, you see that in some sense division is no longer necessary. In enabling division, you have created a system in which every (nonzero) number has a *multiplicative inverse* or *reciprocal.* In this system, division by a number (other than zero) is accomplished by multiplying by its reciprocal, which is the source of the "invert and multiply" rule for dividing fractions.

Box 3-5

Consequences of the Basic Properties: Formulas for the Arithmetic of Fractions

Fraction notation. The fractions 3/2 and $\frac{3}{2}$ are alternative ways of writing $3 \div 2$. For numbers m and n, with m not 0, both n/m and $\frac{n}{m}$ denote $n \div m$. These are not defined when $m = 0$.

Reciprocal of reciprocal. The reciprocal of the reciprocal of a number is the number itself. For example, $\frac{1}{\frac{1}{5}} = 5$; and $\frac{1}{\frac{1}{\frac{2}{3}}} = \frac{2}{3}$. In general, for m and n not zero, $\frac{1}{\frac{1}{\frac{n}{m}}} = \frac{n}{m}$.

Equality. For m and s not zero, $\frac{n}{m} = \frac{t}{s}$ is true exactly when $n \times s = m \times t$.

Addition of fractions. Adding fractions requires that they have a common denominator. Their sum is the fraction whose numerator is the sum of their numerators and whose denominator is the common denominator. For example, $\frac{2}{3} + \frac{4}{5} = \frac{2\times 5}{3\times 5} + \frac{4\times 3}{5\times 3} = \frac{(2\times 5)+(4\times 3)}{3\times 5} = \frac{22}{15}$.

In general, for m and s not zero, $\frac{n}{m} + \frac{t}{s} = \frac{n\times s}{m\times s} + \frac{t\times m}{s\times m} = \frac{(n\times s)+(t\times m)}{m\times s}$.

Multiplication of fractions. The product of two fractions is the fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators. For example, $\frac{2}{3} \times \frac{5}{7} = \frac{2\times 5}{3\times 7} = \frac{10}{21}$. In general, for m and s not zero, $\frac{n}{m} \times \frac{t}{s} = \frac{n\times t}{m\times s}$.

Division of fractions. Dividing by a fraction is the same as multiplying by its reciprocal. For example, $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{2\times 7}{3\times 5} = \frac{14}{15}$. In general, for m, s, and t not zero,
$\frac{n}{m} \div \frac{t}{s} = \frac{n}{m} \times \frac{s}{t} = \frac{n\times s}{m\times t}$.

The Rational Numbers

You have seen how a desire to solve subtraction problems with no solutions in whole numbers leads to the construction of the integers. In a very similar way, the desire to solve division problems with no solutions in whole numbers leads to the construction of the positive rational numbers (along with zero). But neither of these number systems does it all: There are some integers that will not divide a given integer, and there are