

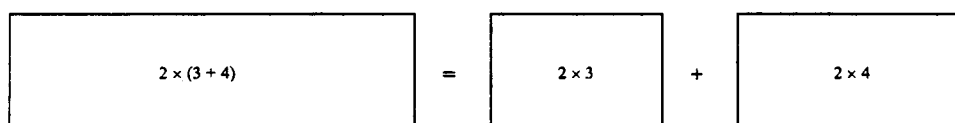
Commutativity and associativity guarantee that all 12 ways of doing this sum give the same answer—so it does not matter which one I do. (For adding 4 numbers, there are 120 (!) conceivable different schemes, all of which again give the same result.) This flexibility is very useful when students come to do computations. For example,  $1 + 8$  can be solved by thinking of it as  $8 + 1$  and then just recalling the next whole number after 8. The standard procedures for doing multidigit arithmetic also heavily exploit commutativity and associativity. However, the flexibility permitted by these rules also greatly increases the challenges of teaching arithmetic. When there are several ways to do a calculation, it is virtually certain that students will produce the answer more than one way. A teacher must therefore have a sufficiently flexible knowledge of arithmetic to evaluate the various student solutions, to validate the correct ones, and to correct errors productively.

The commutative and associative laws also hold for multiplication (see Box 3-1). The commutativity of multiplication by 2 is also reflected in the equivalence of the two definitions of even number typically offered by children. The “fair share” definition says that a number is even if it can be divided into two equal parts with nothing left over (which may be written as  $2 \times m$ ); the “pairing” definition says that a number is even if it can be divided into pairs with nothing left over ( $m \times 2$ ).

In addition to these two laws for each operation, there is a rule, known as the distributive law, connecting the two operations. It can be written symbolically as

$$a \times (b + c) = a \times b + a \times c.$$

An example would be  $2 \times (3 + 4) = 2 \times 7 = 14 = 6 + 8 = 2 \times 3 + 2 \times 4$ . A good way to visualize the distributive law is in terms of the area interpretation of multiplication. Then it says that if I have two rectangles of the same height, the sum of their areas is equal to the area of the rectangle gotten by joining the two rectangles into a single one, of the same height, but with base equal to the sum of the bases of the two rectangles:

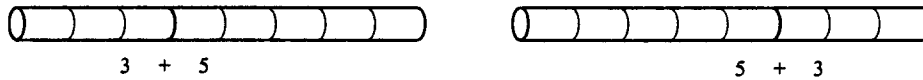


The basic properties of addition and multiplication of whole numbers are summarized in Box 3-1.

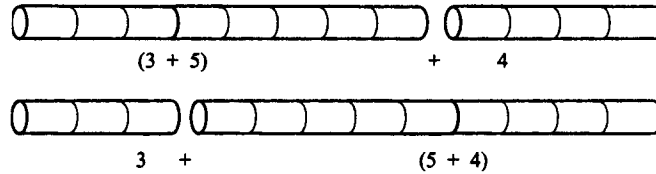
**Box 3-1**

**Properties of the Arithmetic Operations**

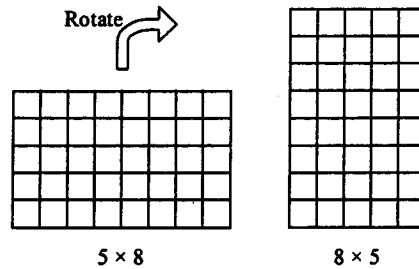
**Commutativity of addition.** The order of the two numbers does not affect their sum:  $3 + 5 = 8 = 5 + 3$ . In general,  $m + n = n + m$ .



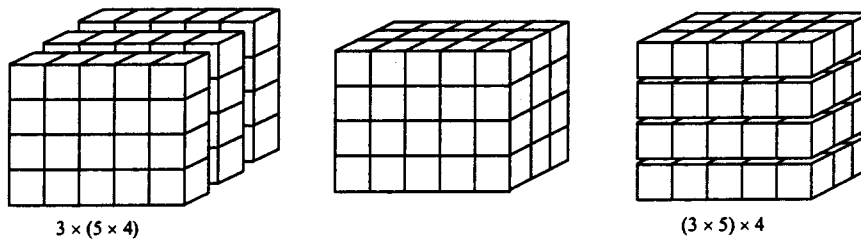
**Associativity of addition.** When adding three (or more) numbers, it does not matter whether the first pair or the last pair is added first:  $(3 + 5) + 4 = 8 + 4 = 12 = 3 + 9 = 3 + (5 + 4)$ . In general,  $(m + n) + p = m + (n + p)$ .



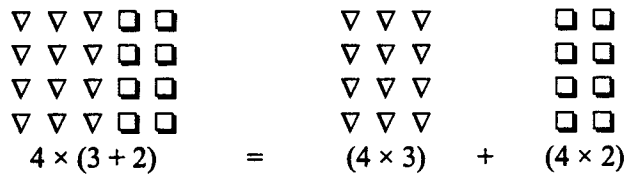
**Commutativity of multiplication.** The order of the two numbers does not affect their product:  $5 \times 8$  produces the same answer as  $8 \times 5$ . In general,  $m \times n = n \times m$ .



**Associativity of multiplication.** When multiplying three or more numbers, it does not matter whether the first pair or the last pair is multiplied first:  $3 \times (5 \times 4)$  is the same as  $(3 \times 5) \times 4$ . In general,  $(m \times n) \times p = m \times (n \times p)$ .



**Distributivity of multiplication over addition.** When multiplying a sum of two numbers by a third number, it does not matter whether you find the sum first and then multiply or you first multiply each number to be added and then add the two products:  $4 \times (3 + 2) = (4 \times 3) + (4 \times 2)$ . In general,  $m \times (n + p) = (m \times n) + (m \times p)$ .



**Question:** Is subtraction commutative?

*Answer:* No. For example,  $6 - 2 = 4$ , but  $2 - 6 = -4$ .

*Question:* Is subtraction associative?

*Answer:* No. For example,  $(7 - 4) - 2 = 3 - 2 = 1$ , but  $7 - (4 - 2) = 7 - 2 = 5$ .

### **Subtraction and Division**

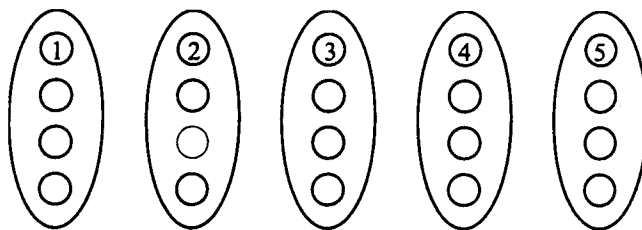
So far we have been talking only about addition and multiplication. It is traditional, however, to list four basic operations: addition and subtraction, multiplication and division. As is implied by the usual juxtapositions, subtraction is related to addition, and division is related to multiplication. The relation is in some sense an inverse one. By this, we mean that subtraction undoes addition, and division undoes multiplication. This statement needs more explanation.

Just as people sometimes want to join sets, they sometimes want to break them apart. If Eileen has eight apples and eats three, how many does she have left? The answer can be pictured by thinking of eight apples as composed of two groups, a group of five apples and a group of three apples. When the three are taken away, the five are left. In this solution, you think of eight as  $5 + 3$ , and then when you subtract the three, you are again left with five. Thus subtracting 3 undoes the implicit addition of three and leaves you with the original amount. It is the same no matter what amount you start with:  $5 + 3 - 3 = 5$ ;  $9 + 3 - 3 = 9$ ;  $743 + 3 - 3 = 743$ . More formally, subtracting three is the *inverse* of adding three.

It is similar with division and multiplication. Just as people sometimes want to form sets of the same size into one larger set, they sometimes want to break up a large set into equal-size pieces. If you think of fifteen as  $5 \times 3$ , then when you divide fifteen by three, you are again left with five. Thus division by three undoes implicit multiplication by three and leaves you with the original amount. It is the same no matter what amount you start with:  $5 \times 3 \div 3 = 5$ ;  $9 \times 3 \div 3 = 9$ ;  $743 \times 3 \div 3 = 743$ . More formally, dividing by three is the *inverse* of multiplying by three.

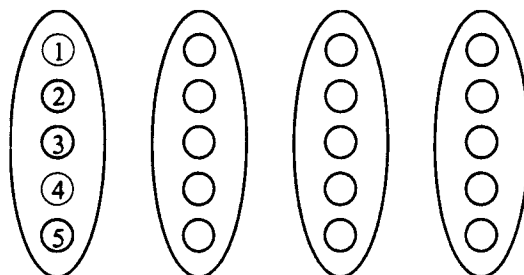
Two interpretations of division deserve particular mention here. If I have twenty cookies, and want to sort them into five bags, how many go in each bag? This is the so-called sharing model of division because I know in how many ways the cookies are to be shared. I can find the answer by picturing the twenty cookies arranged in five groups of four cookies, which will be the contents of one bag. If the cookies originally came out of

five bags of four each, when I put them back into those bags, I will again have four in each. Thus, division by five undoes multiplication by five, or division by five is the inverse of multiplication by five. The picture below shows the sharing model for this situation.



*Sharing 20 cookies among 5 bags*

To think about  $20 \div 5$ , you could also use the measurement model: If I have twenty cookies that are to be packaged in bags of five each, how many bags will I get? In the sharing model (also called the partitioning model or partitive division), you know the number of groups and seek the number in a group. In the measurement model (also called quotative division), you know the size of the groups and seek the number of groups. The circled numbers in the figures above and below illustrate a crucial difference between the two models: the order in which the cookies are placed in bags. In the sharing model, the cookies are dealt into the bags one at a time; in the measurement model, cookies are counted out by complete bags. When you deal with actual cookies, the processes are quite different, but abstractly they are both  $20 \div 5$ . Note that because multiplication is commutative, five bags of four cookies each is the same total number of cookies as four bags of five cookies each. Eventually students come to see the two kinds of division as interchangeable and use whichever model helps them with a particular division problem.



*Measuring 20 cookies into bags of 5 each*

## Subtraction and the Integers

We might summarize the story so far by saying that there are two pairs of operations—addition and subtraction, and multiplication and division—and these are inversely related in the sense described above. However, this summary would not quite be correct. In fact, subtraction is not actually an operation on whole numbers in the same sense that addition is. You can add *any* pair of whole numbers together, and the result is again a whole number. Sometimes, however, you cannot subtract one whole number from another. If I have three apples, and Bart asks for five, I can't give them to him. I just don't have five apples. If I'm really supposed to give him five apples (maybe he left five apples in my care, I ate two, and then he came back to reclaim his apples), then I am in trouble. This situation can be described by using negative numbers: I have negative-two apples, meaning that after I give Bart all the apples I have, I still owe him two more. What is happening mathematically is that I have bumped up against a subtraction problem,  $3 - 5$ , for which there is no solution (in whole numbers). Mathematicians respond by inventing a solution for it, and they call the solution  $-2$ .

Thus, the desire to describe solutions for certain “impossible” subtraction problems leads to the invention of new numbers, the negative integers.<sup>4</sup> Thanks to the negative integers, you can solve *all* whole number subtraction problems. But your problems are not over. You soon find that you cannot be content simply to admire these new creations. You get into situations in which you want to do arithmetic with them also. If I owe Bart two apples, and I owe Teresa four apples, how many apples do I owe all together—that is, what is  $(-2) + (-4)$ ? If on Monday I get into a situation that leaves me two apples short, and this happens again on Tuesday and Wednesday, how many apples short am I then—that is, what is  $3 \times (-2)$ ? Besides enlarging their idea of number, people have had to extend the arithmetic operations to this new larger class of numbers. They have needed to create a new, enlarged *number system*. The new system, encompassing both positive and negative whole numbers, is called the *integers*.

How do people decide what arithmetic in this extended system is (or should be)? How do they create recipes for adding and multiplying integers, and what are the properties of these extended operations? They have two guides: (a) intuition, and (b) the rules of arithmetic, as described above and in Box 3-1. Fortunately, the guides agree.