# Session 9

# Measurement Relationships

## Key Terms in This Session

**Previously Introduced**

- area
- perimeter
- surface area
- volume

## Introduction

In this session, you will explore the dynamic relationships that exist among measurements, such as area and perimeter or surface area and volume. More specifically, you will look at what happens to one variable when the other one is fixed. You will also consider some practical applications of these relationships.

For information on required and/or optional materials, **see Note 1.**

## Learning Objectives

In this session, you will do the following:

- Examine the relationships between area and perimeter when one measure is fixed
- Explore which shapes simultaneously maximize area and minimize perimeter, and vice versa
- Learn about the proportional relationship between surface area and volume and some of its applications
- Construct open boxes and use graphs to approximate the dimensions of a rectangular prism that holds the maximum volume

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**Note 1. Materials Needed:**

- Scissors
- Tape
- Graphing calculator (optional)
- Multilink cubes (3/4 in. cubic units that snap together)
- Square one-inch tiles (optional)

Multilink cubes and square one-inch tiles can be purchased from:

ETA/Cuisenaire, 500 Greenview Court, Vernon Hills, IL 60061; Phone: 800-445-5985/800-816-5050 (Customer service); Fax: 800-875-9643/847-816-5066; http://www.etacuisenaire.com

# Part A: Area and Perimeter (45 min.)

## Constant Perimeter

If you have 72 ft. of fencing and you want to use it to make a rectangular pen for your Highland terrier, you must consider both the perimeter of the pen and its area. What relationships exist between these two measures? Do shapes with the same perimeter have the same area? Let's investigate this situation.

**Problem A1.** Imagine that you want to use all 72 ft. of fencing to make the rectangular pen, that the dimensions of the pen in feet will be whole-number values, and that you want the maximum area for your puppy.

a. What are the dimensions of the possible rectangular pens?

b. What are the areas of these pens?

**Problem A2.** All of the pens have a perimeter of 72 ft., yet the areas of the pens differ. What do you notice about the shapes of the pens with small areas as opposed to those with large areas? What are the characteristics of a shape with the greatest area?

In the above problems, you've seen that when you form the fencing into a long, skinny rectangle, the area is small. But the area increases as the rectangle becomes more square-like, and the greatest area occurs when the fencing is in the shape of a square or square-like rectangle. This leads us to consider shapes other than rectangles. For example, if the perimeter remained the same, would an equilateral triangle or a regular pentagon or a regular hexagon have the same area or more or less area than the square?

**Problem A3.** Imagine the pen were in the shape of an equilateral triangle. What is the area of this triangular pen? **[See Tip A3, page 187]**

### Take It Further

**Problem A4.** Imagine the pen were in the shape of a regular hexagon. What is the area of this hexagonal pen? **[See Tip A4, page 187]**

**Problem A5.** Would other shapes give the puppy even more square footage? Imagine building a circular pen. Find the area when the circumference is exactly 72 ft.

In the activities above, you've seen that when the perimeter is fixed, shapes that have many sides have a greater area. In fact, the shape with the greatest area when the perimeter remains constant is a circle.

**Problem A6.** Imagine you have a barn that is 70 ft. long on your property. You plan to use a part of the existing barn wall as one side of the fence. What are some options for the shape of the pen? What shape of the pen will give the greatest area under these conditions? **[See Tip A6, page 187]**

**Video Segment** (approximate time: 04:56-06:20): You can find this segment on the session video approximately 4 minutes and 56 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

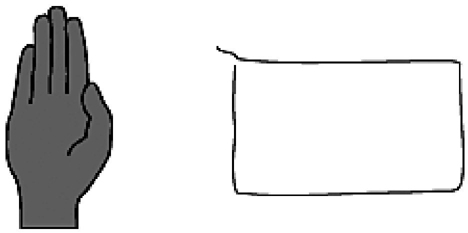
In this video segment, David Cellucci and David Russell examine how to maximize the area of the puppy pen built against a barn. They try several different shapes, including rectangles and semicircles. Watch this segment after you've completed Problem A6.

Were your findings similar?

# Part A, cont'd.

## Constant Area

There are other situations that involve both area and perimeter. Consider this one: Joel, a student in a sixth-grade class, completed an exercise similar to one in Session 6. After tracing his hand on grid paper, he was asked to approximate its area. Instead of counting squares, he took a piece of string and traced the perimeter of his hand. He then took the length of string that represented the perimeter of his hand, reshaped it into a rectangle, and found the area of the rectangle. Joel concluded that the area of his hand and the area of the rectangle were the same.



**Problem A7.** Think about Joel's strategy and conclusion. Will his strategy work in all situations? Do you agree or disagree with his conclusions? **[See Tip A7, page 187]**

Looking at this situation from another direction, do figures with the same area always have the same perimeter? Why or why not? And if not, which perimeters are possible, and which are impossible?

For a non-interactive version of this activity, use 12 one-inch square plastic or ceramic tiles (or Scrabble tiles). Arrange, and rearrange, all 12 square tiles to make plane figures (see below) with an area of 12 square units. Each tile must share at least one side with another tile. Then measure the perimeter of each shape you create.

**Try It Online!** **www.learner.org**

This problem can be explored online as an Interactive Activity. Go to the *Measurement* Web site and find Session 9, Part A.

**Problem A8.**

a. What is the smallest perimeter possible using 12 square tiles?

b. What is the largest possible perimeter?

**Problem A9.**

a. Make figures with an area of 12 square units with perimeters of 14 through 26 units. Keep in mind that it is not possible to create all of these perimeters. Sketch the shapes and record their areas and perimeters.

b. Choose one perimeter between 14 and 26 that you could not make and explain why it is impossible.

**Problem A10.** Under what circumstances might you want the smallest perimeter for a set area?

# Part B: Surface Area and Volume (40 min.)

## Determining the Relationship

In the previous part, you learned that the relationship between perimeter and area is dynamic; namely, the amount of area or perimeter of a shape is not fixed in relation to the measure of the other variable.

How about the relationship between surface area and volume? Do prisms with the same volume have the same surface area? Let's explore this relationship.

**Problem B1.** Take 24 multilink cubes or building blocks and imagine that each cube represents a fancy chocolate truffle. For shipping purposes, these truffles need to be packaged into boxes in the shape of rectangular prisms. Knowing that you must always package 24 truffles (i.e., your volume is set at 24 cubic units), what are the possible dimensions for the boxes? Record your information in the table below: **[See Tip B1, page 187]**

| Length | Width | Height | Volume | Surface Area |
|---|---|---|---|---|
| | | | 24 | |
| | | | 24 | |
| | | | 24 | |
| | | | 24 | |
| | | | 24 | |
| | | | 24 | |

**Problem B2.** Which of your packaging arrangements requires a box with the least amount of material? The greatest amount of material? Why is the amount of material needed for packaging important?

**Problem B3.** What do you notice about the shape of the package that has the smallest surface area? How about the package with the greatest surface area?

When the volume is constant (as in the truffles problem), the surface area depends on the shape of the solid. But what happens to the surface area of a solid as its volume increases—does surface area increase at the same rate as volume?

**Problem B4.** Use multilink cubes or building blocks to create different-sized cubes from the table below. Calculate the volumes and surface areas of the cubes. Examine the proportional relationship between surface area (SA) and volume (V) by creating a surface area-to-volume ratio (SA:V). As volume increases, what happens to the ratio of surface area to volume? **[See Tip B4, page 187]**

| Size of Cube | Surface Area | Volume | Ratio SA:V |
|---|---|---|---|
| 1 by 1 by 1 | | | |
| 2 by 2 by 2 | | | |
| 3 by 3 by 3 | | | |
| 4 by 4 by 4 | | | |
| 5 by 5 by 5 | | | |
| 6 by 6 by 6 | | | |
| 7 by 7 by 7 | | | |
| 8 by 8 by 8 | | | |

# Part B, cont'd.

**Video Segment** (approximate time: 11:49-13:18): You can find this segment on the session video approximately 11 minutes and 49 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Watch this video segment after you've completed Problem B4 to see what Lori and Jayne found out about the relationship between the volume and surface area of solid objects.

Did you express this relationship differently?

**Problem B5.** Over the last decade, we have seen the genesis of the giant "superstore." What kind of surface area-to-volume ratio do you think superstores have? Why do companies build such large stores?

## Human Measurements

Your body's surface area is a measurement of the skin that covers your body. You may have noticed that adults and children (and babies in particular) have very different reactions to heat and cold. This happens because the body cools down by sweating at a rate proportional to the area of its skin, but warms up in proportion to its mass (volume). The ratio of surface area to mass is much larger for babies, so they cool down faster than adults. As a result, babies can catch a chill even when adults feel warm.

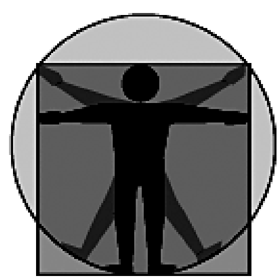
**Problem B6.** Use multilink cubes to build a model of a baby and of an adult. You can use a simple, trimmed-down model, or you can build a more realistic one. Once you've completed your models, calculate the surface area-to-volume ratio for the baby and the adult.

**Problem B7.** In the summer, we're warned not to leave babies or pets in cars. Yet on a hot day, an adult can sit in a car for a short period of time without harm. Use your models and mathematics to explain what is occurring in these situations, and why babies dehydrate so much more quickly than adults.

### Take It Further

**Problem B8.** In Session 6, you learned that about 100 handprints will cover the body. You used your estimate of the area of your handprint to approximate your surface area. Another approach to estimating surface area is to see how much of you fits into a square.

The picture at right is based on a famous drawing by Leonardo da Vinci. As shown in the picture, the person more or less fits in the square.



A person's height is approximately equal to his or her arm span (from fingertip to fingertip).

a. Measure your height and arm span in centimeters and find the area of "your square."

b. Many have suggested that three-fifths of the square is a reasonable approximation for surface area. How does three-fifths of the area of your square compare with your first approximation of your surface area based on hand size?

**Problem B9.** Another way to approximate a person's surface area is to use a simple formula:

Height (cm) • Thigh Circumference (cm) • 2 = Body Surface Area ($cm^2$)

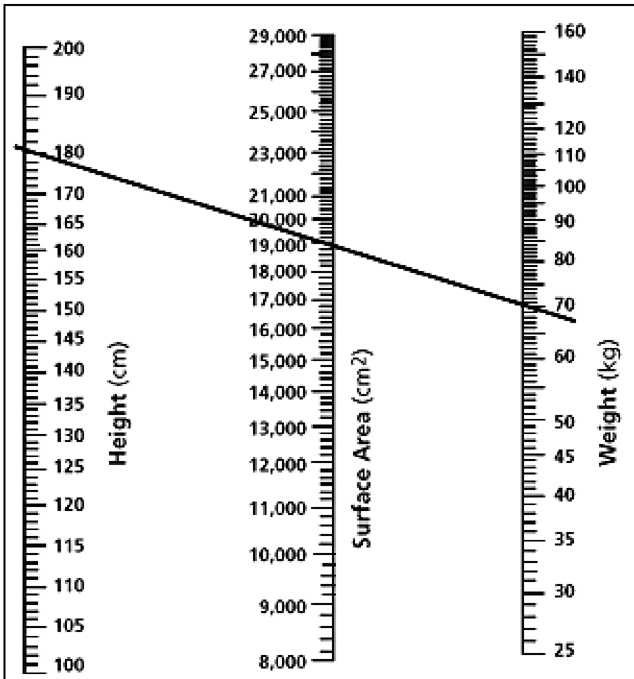
a. Find your surface area using this formula.

b. What do you think the above formula is based on?

c. Do you think the above formula would work for determining the surface area of a baby? Why or why not?

Since surface area is also related to weight, health care workers usually use a chart called a nomograph to estimate a person's surface area. To use the nomograph, a person's height (in centimeters) is located in the left-hand column, and a person's weight (in kilograms) is located in the right-hand column. These points are connected with a straight line. The surface area of a person's body is shown where the line crosses the middle scale. **[See Note 2]**

**Problem B10.** Use the nomograph to estimate your own body's surface area.
Note that 1 kg = 2.2 lb.




Nomograph for Estimating Body Surface Area from Height and Weight

**Problem B11.** The density of the human body is a little greater than the density of water because of our bones and organs. One kilogram of body mass occupies a volume of about 0.9 L. Determine the volume of your body by multiplying your weight in kilograms by 0.9.

**Problem B12.**

a. What is your surface area-to-volume ratio?

b. What is the surface area-to-volume ratio of a child who weighs 55 lb. and is 40 in. tall?

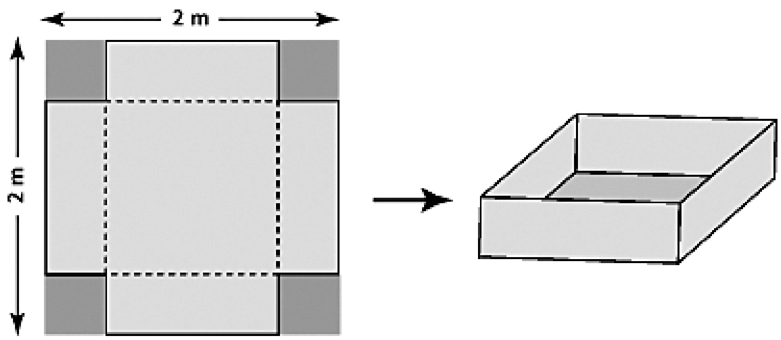
c. Compare the two ratios. How do these measurements compare with your first estimates?

**[See Tip B12, page 187]**

**Note 2.** The problems in this section require some conversions between the U.S. customary system and the metric system. You can use the Web site www.onlineconversion.com to find the proper conversions.

# Part C: Designing a Water Tank (35 min.)

Imagine constructing an open-topped water tank from a square metal sheet (2 m by 2 m). You would cut squares from the four corners of the sheet and bend up the four remaining rectangular pieces to form the sides of the tank. Then you would weld the edges together to make them watertight. Your goal is to construct a tank with the greatest possible volume.




What size squares do you conjecture would result in the water tank with the greatest volume?

To investigate the relationship between the maximum volume of the tank and the size of the squares cut from the corners, build models and collect data. Using a 1:10 scale, start with a model—use the 20-by-20-cm square sheet of paper on pages 184-185 and make multiple copies of it. Take one sheet of paper and cut a 1-by-1-cm square from each corner. Fold the net into an open box and tape it. You have just constructed a scale model of the water tank. Repeat the process to construct different models of the water tank.

**Problem C1.** Collect data on the different-sized water tanks you can make. The side lengths of the cutout squares in centimeters must be whole-number values. Record your data by filling in the table below:

| Size of the Cutout Square (cm) | Dimensions of the Box (cm) | Volume of the Box ($cm^3$) |
|---|---|---|
| 1 by 1 | 1 by 18 by 18 | 324 |
| | | |
| | | |
| | | |
| | | |

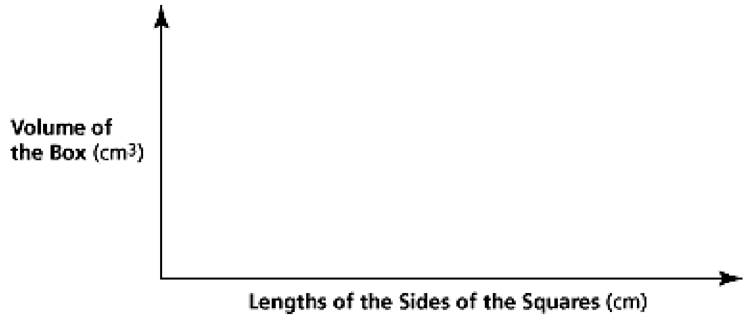
**Problem C2.** Using whole-number side lengths, which size of a cutout square results in the largest volume for the box? What is the size of the cutout square and the resulting volume?

**Problem C3.** These models were at a scale of 1:10. If the largest box you made represented the water tank, what would its dimensions be?

"Designing a Water Tank" adapted from Swan, Malcolm, and the Shell Centre Team. *The Language of Functions and Graphs.* p. 146. ©1999 by Shell Centre Publications. http://www.MathShell.com.

# Part C, cont'd.

**Problem C4.** One way to get a closer estimate of the dimensions of the box with the greatest volume is to make a graph. On the x-axis, plot the length of the sides of the cutout square (in centimeters); on the y-axis, plot the corresponding volume of the box (in cubic centimeters). Use a graphing calculator, sketch your graph on the grid paper on page 186, or enter the data into the graphing program on your computer.




## Take It Further

**Problem C5.** What would happen if you could remove squares from the corners that used decimals, such as side of square = 3.5 cm, or side = 3.75 cm? Approximate the size of the squares that should be cut to maximize the resulting volume. **[See Note 3]**

**Video Segment** (approximate time: 18:03-20:22): You can find this segment on the session video approximately 18 minutes and 3 seconds after the Annenberg/CPB logo. Use the video image to locate where to begin viewing.

Find out how the course participants went about modeling a container with the maximum volume by cutting out different-sized squares from the 20-by-20 sheet of paper. Jayne and Lori try to use one of their earlier observations about surface area and volume. David Russell and David Cellucci graph the data, which leads to new observations. They consider the effect of the absence of a lid on a relationship between volume and surface area.

Why do you think Jayne and Lori's initial intuitive approach didn't work? How would you explain it in your own words?

**Note 3.** Graphing the data may help you realize that the side length of the square that produces the maximum volume is not a whole-number value. Also, while we can approximate the maximum volume of the water tank, the actual maximum is not easily determined by this method. To find the maximum volume more precisely, we would need to use calculus.

# Homework

**Problem H1.** Reptiles need the sun to get them going. Many large dinosaurs had to have large fins or plates down their back. How does this fact relate to what you have learned about the proportion of surface area to volume?

## Take It Further

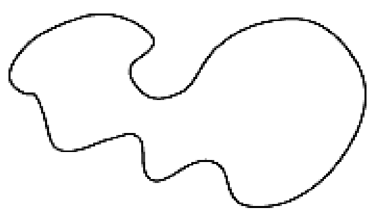
**Problem H2.** A manufacturer is producing half-liter aluminum cans in a cylindrical shape. The volume of the can is 500 $cm^3$.

a. Find the radius and height for the can that will use the least aluminum and therefore be the cheapest to manufacture. In other words, minimize the surface area of the can.

b. What shape is your can? Do you know of any cans that are made in this shape? Can you think of any practical reasons why more cans are not made in this shape?

**Problem H3.** Some aspirin-like tablets are said to work "two and a half times faster" than their competitors. What is an obvious way in which this could be accomplished?

**Problem H4.** Mr. Hobbs had an ugly blob in the middle of his wall. The paint on the rest of the wall looked fresh, so Mr. Hobbs asked the painter to come and paint only the blob. The painter said he would charge according to the area that needed to be painted. To figure out how much the job would cost, Mr. Hobbs ran a string around the edge of the blob so that it covered the border perfectly. Then, to figure out the area, he removed the string, shaped it into a rectangle, and figured out the area of the rectangle.



How close did Mr. Hobbs's estimate come to the painter's bill for painting the blob?

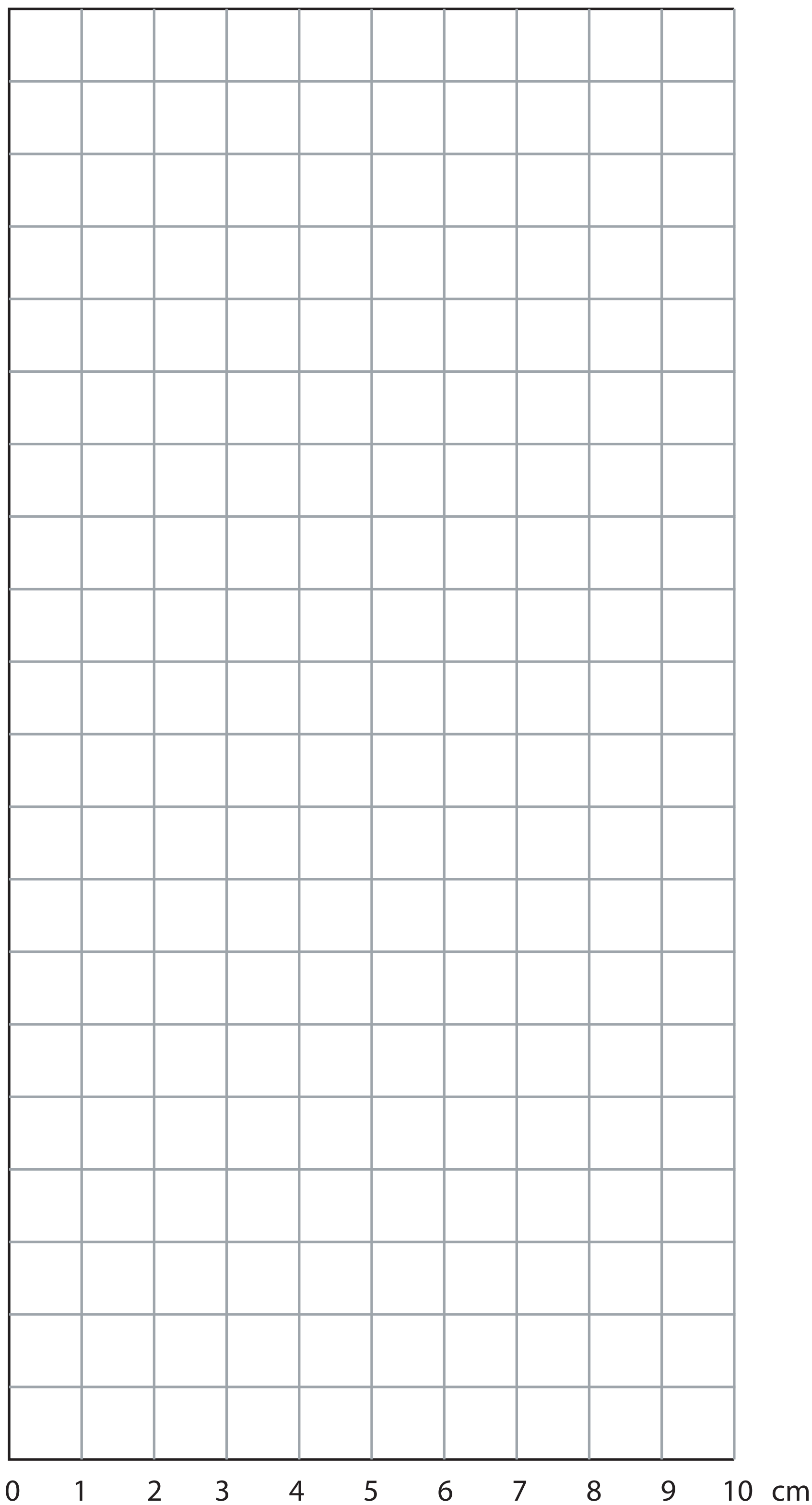
**Problem H5.** In cubes, you found a proportional relationship between volume and surface area. Does the proportional relationship between volume and surface area also exist for spheres? **[See Tip H5, page 187]**

Problem H2 adapted from Swan, Malcolm, and the Shell Centre Team. *The Language of Functions and Graphs.* p. 174. ©1999 by Shell Centre Publications. http:// www.MathShell.com.

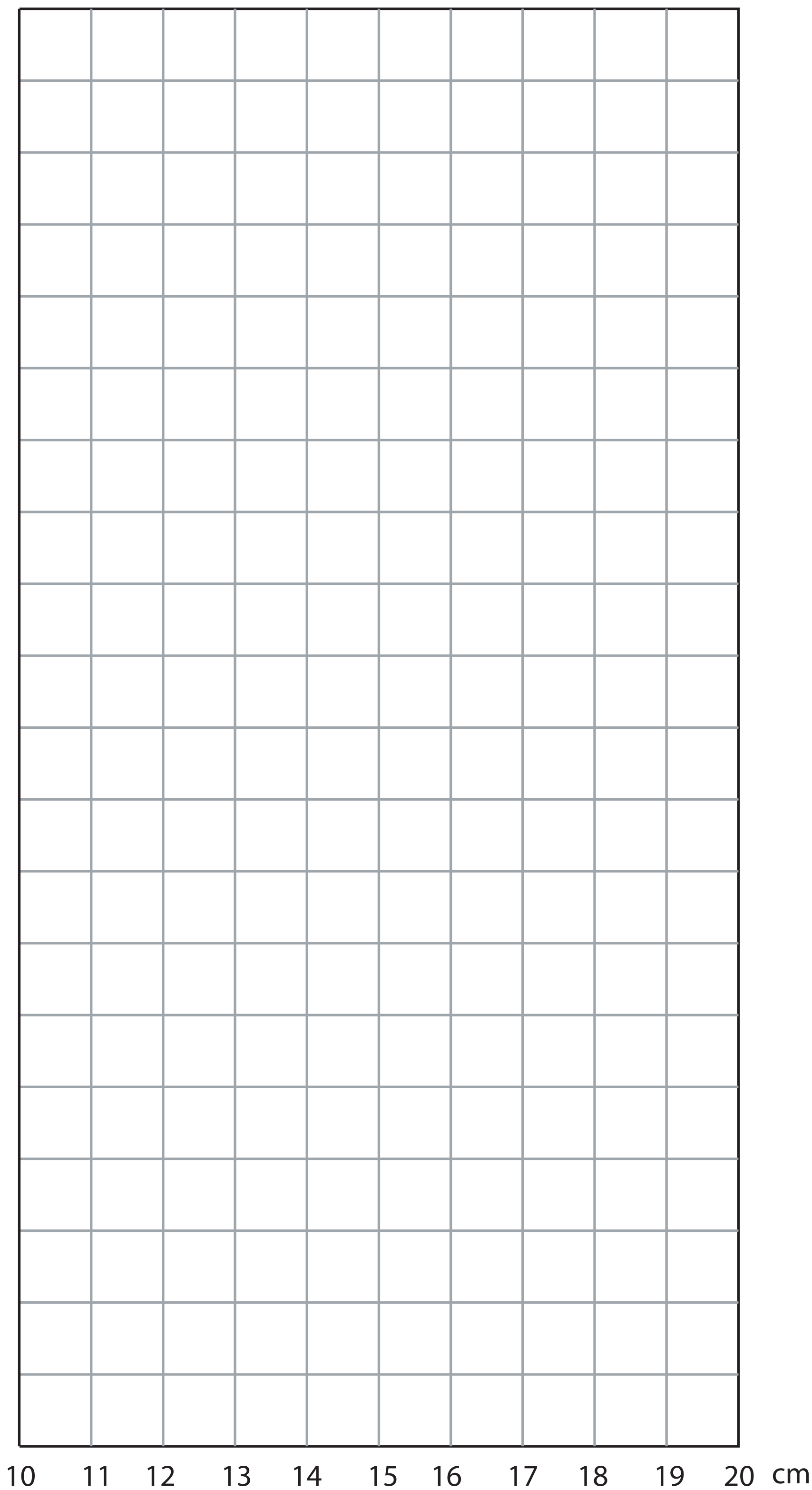
Problem H4 adapted from Lamon, Susan J. *Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies for Teachers*. ©1999 by Lawrence Erlbaum Associates.

## Grid for Problem C1, page 1

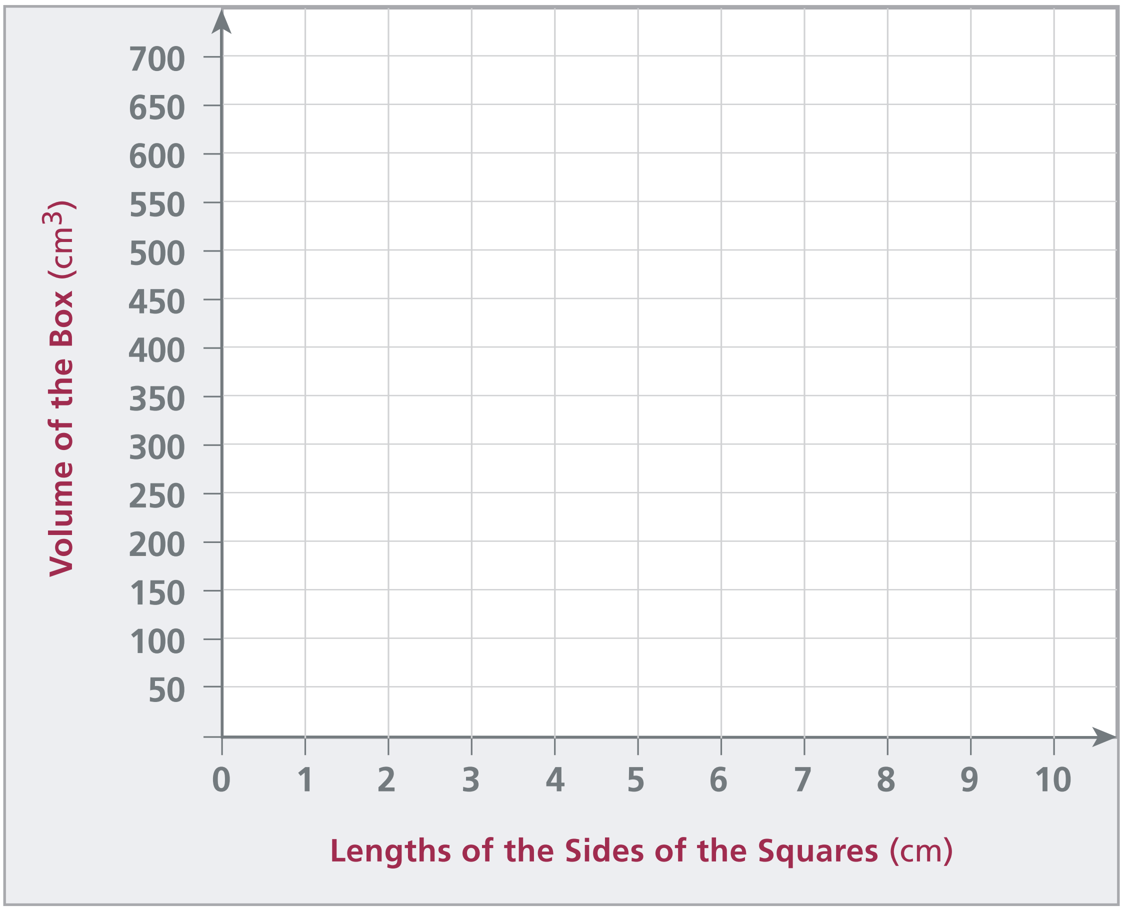
Print out both pages and tape together to create a 20 x 20 cm square grid.



**Grid for Problem C1, page 2**



**Graph for Problem C4**



# Tips

## Part A: Area and Perimeter

**Tip A3.** Draw a picture of a triangle. How long is each side? Draw the height of the triangle, and use the Pythagorean theorem to find the height.

**Tip A4.** Draw a picture of a hexagon. What is the length of each side? You can divide the hexagon into six equilateral triangles and calculate the area of a single triangle using the method from Problem A3.

**Tip A6.** Consider several different shapes for the pen—square, rectangle, semicircle—and collect data for each of them.

**Tip A7**. Think about what you discovered in Problems A1 and A2 and how that information might be used to analyze Joel's strategy.

## Part B: Surface Area and Volume

**Tip B1.** The dimensions must all be factors of 24—1, 2, 3, 4, 6, 8, 12, 24.

**Tip B4**. You may find it helpful to build the different-sized cubes first and then to use the models to confirm your calculations of the volumes and surface areas. If cubes aren't available, you could make a sketch of the cubes on graph paper to help you visualize the surface area of each face. Write the surface area-to-volume ratios and look for patterns.

**Tip B12.** This problem will be easier to solve if you convert to metric measures.

## Homework

**Tip H5.** The volume of a sphere is $(4/3)\pi r^3$, and the surface area of a sphere is $4\pi r^2$.

# Solutions

## Part A: Area and Perimeter

**Problem A1.**

a. There are many possibilities, but the length and width must add up to 36 ft., since the perimeter (made up of two lengths and two widths) is 72 ft.

b. Possible areas go from 35 $ft^2$ (35 by 1) to 324 $ft^2$ (18 by 18).

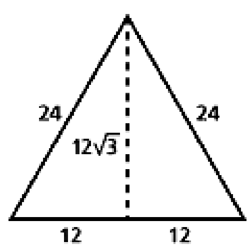
**Problem A1 Table**

| Length | Width | Area |
|---|---|---|
| 1 | 35 | 35 |
| 2 | 34 | 68 |
| 3 | 33 | 99 |
| 4 | 32 | 128 |
| 5 | 31 | 155 |
| 6 | 30 | 180 |
| 7 | 29 | 203 |
| 8 | 28 | 224 |
| 9 | 27 | 243 |
| 10 | 26 | 260 |
| 11 | 25 | 275 |
| 12 | 24 | 288 |
| 13 | 23 | 299 |
| 14 | 22 | 308 |
| 15 | 21 | 315 |
| 16 | 20 | 320 |
| 17 | 19 | 323 |
| 18 | 18 | 324 |
| 19 | 17 | 323 |
| 20 | 16 | 320 |
| 21 | 15 | 315 |
| 22 | 14 | 308 |
| 23 | 13 | 299 |
| 24 | 12 | 288 |
| 25 | 11 | 275 |
| 26 | 10 | 260 |
| 27 | 9 | 243 |
| 28 | 8 | 224 |
| 29 | 7 | 203 |
| 30 | 6 | 180 |
| 31 | 5 | 155 |
| 32 | 4 | 128 |
| 33 | 3 | 99 |
| 34 | 2 | 68 |
| 35 | 1 | 35 |

**Problem A2.** The shape with the largest area for this particular perimeter is the square 18 by 18 ft. In general, figures whose length and width are close to one another have larger areas than figures whose length and width are very different.

**Problem A3.** If the pen were triangular, and its perimeter equal to 72 ft., the length of each side would be 72 ÷ 3 = 24 ft. So the area is A = (1/2)*bh*. Using the Pythagorean theorem to calculate the height (*h*), we'd get the following:

$$h = \sqrt{24^2 - 12^2} = \sqrt{576 - 144} = \sqrt{432} = \sqrt{3 \cdot 144} = 12\sqrt{3}.$$



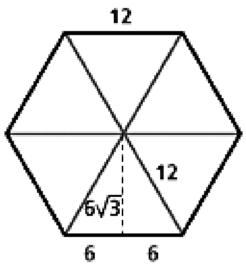

So the area is:

$$A = \frac{24 \cdot 12\sqrt{3}}{2} = 144\sqrt{3} \approx 249.4 \text{ ft.}$$

This is significantly less than the areas of some of the rectangular pens in Problem A1.

# Solutions, cont'd.

**Problem A4 .** If the pen were hexagonal, and its perimeter equal to 72 ft., the length of each side would be $72 \div 6 = 12$ ft. The area will be six times the area of the equilateral triangle inside the hexagon (see picture).




So the area is:

$$A = 6 \cdot A_{Triangle} = 6 \cdot \left(\frac{12 \cdot 6\sqrt{3}}{2}\right) = 216\sqrt{3} \approx 374.1 \text{ ft.}$$

As you see, the area of the hexagon is larger than that of any of the rectangular shapes in Problem A1.

**Problem A5.** Since we know that the circumference of the pen is C= 72 ft., and that $C = 2\pi r$, we can calculate the radius, $r = C \div 2\pi = 72 \div 2\pi$, or approximately 11.46 ft. Its area is $\pi \cdot (11.46)^2$, or about 412.53 $ft^2$. This is quite a bit roomier than the largest rectangular pen that could be built.

**Problem A6.** In this situation, the fencing completes three sides of a rectangular pen, and the existing barn wall completes the fourth side. So the largest square you could enclose would have side length 24 ft. and area 576 $ft^2$. The largest rectangle would be 18 by 36, with an area of 848 $ft^2$, larger in area than the square. (Note that because the fencing is not enclosing all four sides, a square does not enclose the largest area in this situation—in fact, the largest area for a rectangular pen occurs when the fencing makes three sides of "half a square," with length equal to twice the width).

A semicircle would have radius $72 \div \pi$, or approximately 22.92 ft. (since $C = 2\pi r$ and half the circumference is equal to $\pi r$). The area of the semicircle is $(1/2) \cdot \pi \cdot (22.92)^2$, or about 825 $ft^2$. This is larger in area than any square or rectangular pen.

**Problem A7.** No, this strategy does not work. As you saw in Problems A1-A6, figures with a constant perimeter can have varied areas depending on the shape of the figure. Because the shapes of Joel's hand and the rectangle are inherently different, it is unlikely that the areas will be the same, even though he may be able to construct the rectangle that approximates the same area. The method he used for determining the perimeter would also have been affected by whether or not he had his fingers spread open.

**Problem A8.**

a. The smallest possible perimeter is 14 units and is found by using a 3-by-4 rectangle of tiles.

b. The largest possible perimeter is 26 units. Many different shapes have this perimeter, but the simplest example is a 1-by-12 rectangle. A large L-shaped piece and a snaking piece where the squares are laid end to end will also have this maximum perimeter.

# Solutions, cont'd.

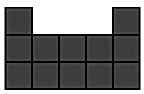
**Problem A9.**

a. The possible perimeters are all even numbers between 14 and 26. Some possibilities:

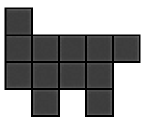
   14 units: a 3-by-4 rectangle

   16 units: a 2-by-6 rectangle

   18 units: a 2-by-5 rectangle with two extra pieces added, which don't touch one another:



   20 units: a 2-by-4 rectangle with four extra pieces added, none of which touch one another:



   22 units: a 2-by-3 rectangle with a row of six extra pieces added:

   24 units: a 2-by-2 square with a row of eight extra pieces added:

   26 units: a row of 12 pieces

b. When pieces border, two segments of perimeter are removed at once. This means that each time the pieces touch, the potential total perimeter remains an even number at all times. No odd number may be the perimeter.

**Problem A10.** Some examples include a room that can be painted as efficiently as possible, or a packing box that can be made out of the least amount of cardboard. You may also want to build the pen for your puppy with the least amount of fencing.

## Part B: Surface Area and Volume

**Problem B1.** Here is the completed table:

| **Length** | **Width** | **Height** | **Volume** | **Surface Area** |
|---|---|---|---|---|
| 1 | 1 | 24 | 24 | 98 |
| 1 | 2 | 12 | 24 | 76 |
| 1 | 3 | 8 | 24 | 70 |
| 1 | 4 | 6 | 24 | 68 |
| 2 | 2 | 6 | 24 | 56 |
| 2 | 3 | 4 | 24 | 52 |

# Solutions, cont'd.

**Problem B2.** The least amount of material is required by the 2-by-3-by-4 box. The greatest amount of material is required by the 1-by-1-by-24 box. The amount of packaging material needed is important, since charges for such material are figured in square inches or square feet, and reducing the amount of needed material will reduce the cost of shipping. Also, the cost of making the box itself will be lower if there is a smaller surface area.

In practical terms, however, you may want to use more packaging material so that the truffles don't get damaged. Also, customers may feel that they're getting more for their money if there is more packaging and the box is larger.

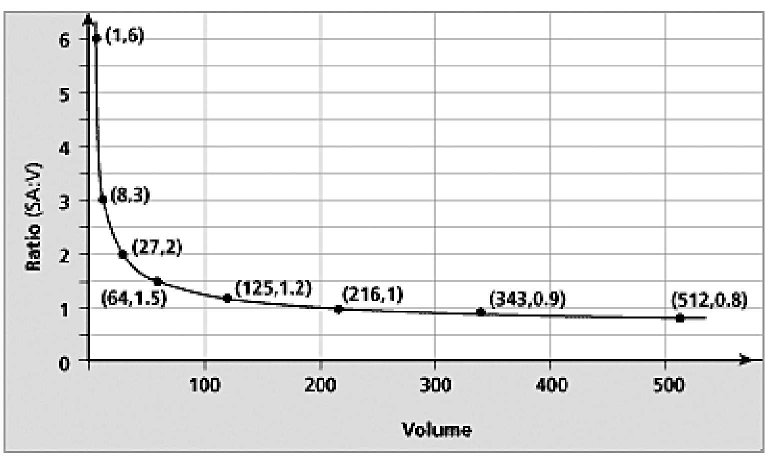
**Problem B3.** The rectangular prisms with the greatest surface area have dimensions that are far apart (1 by 1 by 24), whereas the prisms with smaller surface area have dimensions close to each other (e.g., 2 by 3 by 4).

**Problem B4.** Here is the completed table:

| Size of Cube | Surface Area | Volume | Ratio SA:V |
|---|---|---|---|
| 1 by 1 by 1 | $6 \cdot 1^2 = 6$ | $1^3 = 1$ | 6:1 |
| 2 by 2 by 2 | $6 \cdot 2^2 = 24$ | $2^3 = 8$ | 6:2 (or 3:1) |
| 3 by 3 by 3 | $6 \cdot 3^2 = 54$ | $3^3 = 27$ | 6:3 (or 2:1) |
| 4 by 4 by 4 | $6 \cdot 4^2 = 96$ | $4^3 = 64$ | 6:4 (or 3:2) |
| 5 by 5 by 5 | $6 \cdot 5^2 = 150$ | $5^3 = 125$ | 6:5 |
| 6 by 6 by 6 | $6 \cdot 6^2 = 216$ | $6^3 = 216$ | 6:6 (or 1:1) |
| 7 by 7 by 7 | $6 \cdot 7^2 = 294$ | $7^3 = 343$ | 6:7 |
| 8 by 8 by 8 | $6 \cdot 8^2 = 384$ | $8^3 = 512$ | 6:8 (or 3:4) |

In general, the ratio is $6{:}s$, since the surface area formula is $6s^2$ and the volume formula is $s^3$.

Here is the graph that shows what happens to the ratio of surface area to volume (expressed as a decimal) as the volume increases:




**Problem B5.** The superstore's surface area-to-volume ratio is as small as possible, compared to smaller stores. This gives a company the ability to put more products inside the store (since volume is relatively large), in comparison to how much it may cost to build the store (since surface area is relatively small).

**Problem B6.** Answers will vary, but the baby has a higher surface area-to-volume ratio than the adult.

# Solutions, cont'd.

**Problem B7.** Since babies have a higher surface area-to-volume ratio, and water is a volume-based measurement, babies lose proportionately more water per minute than adults do. As a result, babies will dehydrate more quickly.

**Problem B8.**

a. Answers will vary. The measurements of your height and arm span should be pretty close to each other.

b. Answers will vary, but the measurements should be pretty close.

**Problem B9.**

a. Answers will vary.

b. The formula might be based on empirical measurements only, but it treats the entire body as though it had the same circumference as the thighs. It is somewhat similar to the surface-area formula for a cylinder.

c. It might, but if babies' thighs are not in the same proportions as adults' thighs (and generally they aren't), then this isn't a good measure.

**Problem B10.** Answers will vary. One example is someone with the height of 170 cm and weight of 65 kg, who has the approximate surface area of 17,600 $cm^2$.

**Problem B11.** Answers will vary. For example, a person with the weight of 65 kg will have the volume of 58.5 L.

**Problem B12.**

a. Answers will vary. Using our previous example, the surface area-to-volume ratio is 17,600:58.5.

b. Converting the measures to the metric system, the child's height is about 101.6 cm, and the weight is about 25 kg. Using the nomograph, the surface area is about 8,000 $cm^2$. The volume is 22.5 L. The surface area-to-volume ratio is 8,000:22.5.

c. Converting the two ratios into decimals, we get the ratio for an adult to be about 300.86 and for the child 355.56. As you might expect, the child's surface area-to-volume ratio is higher than an adult's.

## Part C: Designing a Water Tank

**Problem C1.**

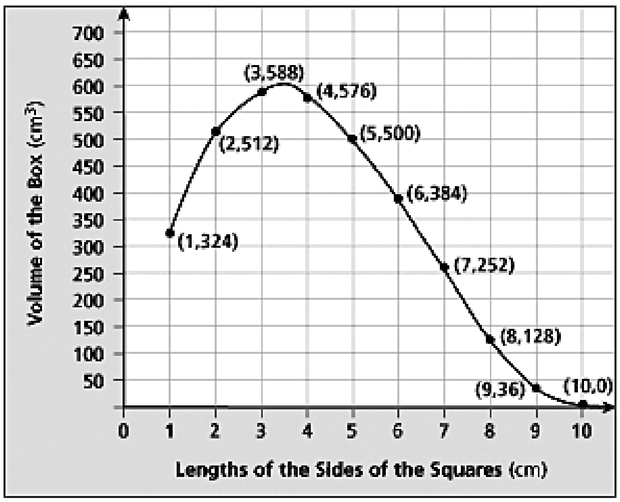
| Size of the Cutout Square (cm) | Dimensions of the Box (cm) | Volume of the Box ($cm^3$) |
|---|---|---|
| 1 by 1 | 1 by 18 by 18 | 324 |
| 2 by 2 | 2 by 16 by 16 | 512 |
| 3 by 3 | 3 by 14 by 14 | 588 |
| 4 by 4 | 4 by 12 by 12 | 576 |
| 5 by 5 | 5 by 10 by 10 | 500 |
| 6 by 6 | 6 by 8 by 8 | 384 |
| 7 by 7 | 7 by 6 by 6 | 252 |
| 8 by 8 | 8 by 4 by 4 | 128 |
| 9 by 9 | 9 by 2 by 2 | 36 |

# Solutions, cont'd.

**Problem C2.** The largest volume seems to result from a 3-by-3 cutout square (588 $cm^3$). The 4-by-4 square gave nearly as high a volume.

**Problem C3.** You found that the largest tank would result if you removed 3-by-3 cm squares. The dimensions of the model would be 17 by 17 by 3 cm. Increasing back to the original scale, the dimensions of the tank would be 170 by 170 by 30 cm.

**Problem C4.**




From observing the graph, it becomes evident that the largest value for volume will be between values 3 and 4 on the x-axis.

**Problem C5.** Using 3.5 as a square's side would give us the volume of 591.5 $cm^3$. Using 3.4 as a square's side, we'd get the volume of 592.4 $cm^3$. The largest volume is achieved when the square is cut with side length 3 1/3 (or 3.333...) cm, leaving 13 1/3 (or 13.333...) cm in the center. The volume is (3 1/3) • (13 1/3) • (13 1/3) = (10/3) • (40/3) • (40/3) = 16,000/27 $cm^3$, or about 592.59 $cm^3$.

# Homework

**Problem H1.** Dinosaurs' plates were likely the result of their very low surface area-to-volume ratio: The plates served to increase the dinosaurs' surface area without increasing their volume very much.

Animals with a great deal of surface area but little volume cool down and heat up faster than animals with larger volumes. This is important in reptiles since they obtain their body heat from the sun.

**Problem H2.**

a. The minimum surface area occurs when the height is exactly twice the radius. If the volume is 500 $cm^3$, the radius is approximately 4.30 cm, while the height is approximately 8.61 cm.

b. In this shape, the can fits perfectly inside a cube, since the diameter of the can is the same as the height. Some cans are made in this shape. When cans are displayed in stores or placed on shelves, there is often more of a premium on radius (shelf space), so the radius is often smaller than our "ideal" can.

**Problem H3.** One obvious way is to increase the surface area-to-volume ratio of the tablet. It would dissolve more quickly. A flat caplet dissolves more quickly than a spherical pill.

# Solutions, cont'd.

**Problem H4.** This is the same type of problem as the "area of the hand" problem in Problem A7. The answer depends on the shape of the rectangle Mr. Hobbs used, since different rectangles with the same perimeter will have different areas. There is no way of guaranteeing that, using this method, the area of the two figures would be the same.

**Problem H5.** Yes, a similar ratio exists. The ratio for spheres is $r$:3 rather than $s$:6 for cubes and can most easily be found by using a table or by dividing the two formulas into one another.