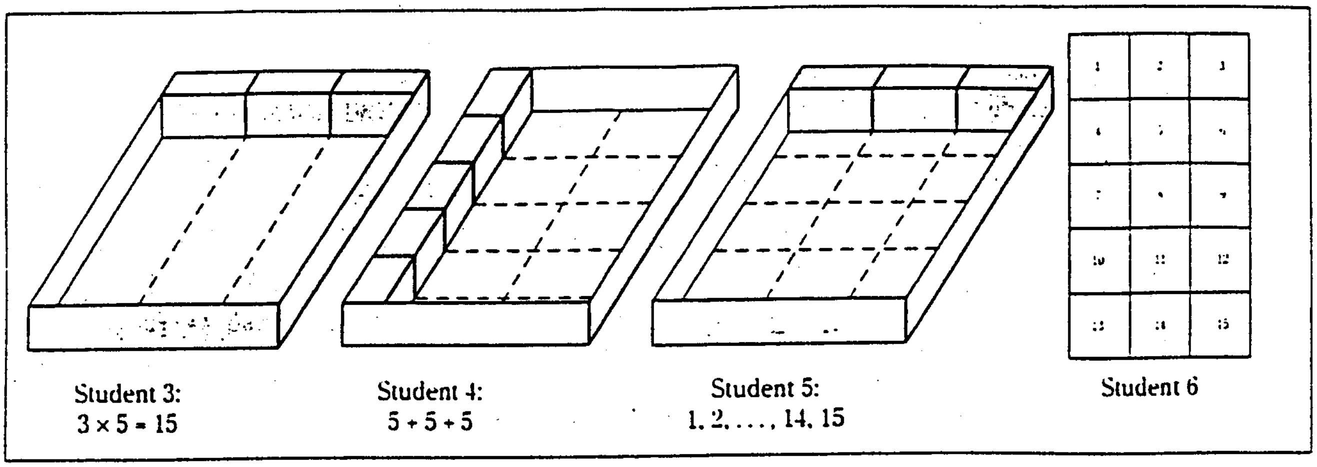


Fig. 6 Student strategies for two-cube-package problem

by the length, which is 5, and came up with 9 times 5, which is 45.

Observer: How do you know that is the right answer?

Clarissa: Because the equation for the volume of a box is length times width times height.

Observer: Do you know why that equation works?

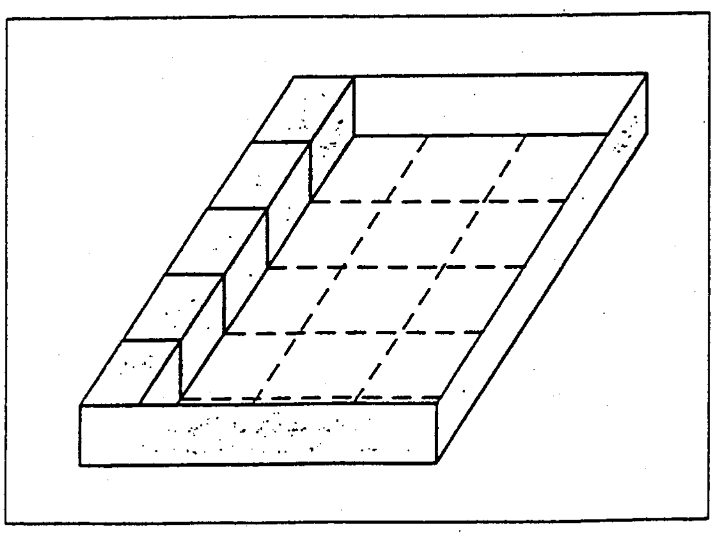
Clarissa: Because you are covering all three dimensions, I think. I'm not really sure. I just know the equation.

Thus, even students with substantial mathematical backgrounds often do not pay sufficient attention to spatial structuring.

Numerical Procedures with Insufficient Structuring

WE NEXT ILLUSTRATE THAT STUDENTS' LACK OF attention to spatial structuring can cause them to attain a superficial understanding of the use of multiplication in certain extremely important geometric contexts. Fifth grader Bethany regularly determined the number of cubes in three-dimensional arrays by using layers. She had also discovered that the number of cubes could be found by multiplying the three dimensions. But as her class discussed these strategies, Bethany raised a concern. When the three dimensions are multiplied, a corner cube gets counted three times; therefore, the method should be wrong: "The corner cube gets counted once when you find the length, once for the width, and once for the height." Even though almost every student in the class had discovered, and was routinely employing, a layering approach, not one of the students had an answer for Bethany's question. Even when the teacher posed Bethany's question in the context of area, the students had no answer.

Fig. 7 Four columns of five packages



The teacher had pairs of students work on the problem. When an observer asked Bethany and her partner how they were thinking about the problem, they said that they were stuck. So the observer posed a question that he thought might help the students clarify their thinking.

Observer: [Arranges a 3-by-3 set of cubes that the students had been working with into 3 rows of 3, and points successively to the cubes in one row] 1, 2, 3. What am I counting here?

Partner: Cubes.

Bethany: Yeah.

Observer: [Pointing to the three rows] 1, 2, 3. What am I counting here?

Bethany: [Excitedly] Rows of cubes. You're not counting cubes this time. So, first, you count cubes, then you count rows.

Partner: So you're not really counting the cube twice. We got it!

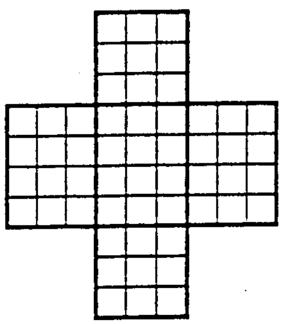
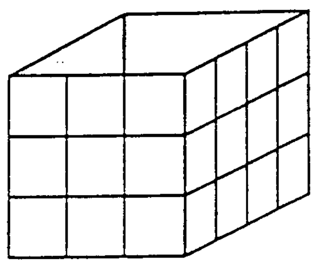
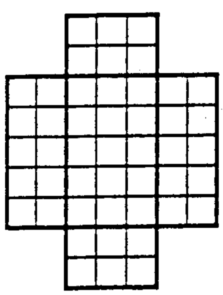
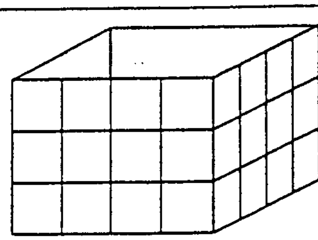
Bethany's question posed a real conundrum for the students. They knew that multiplying the length times the width gave the number of cubes in a rectangular array. Almost all the students justified this procedure by saying that they were multiplying the number of cubes in a row times the number of rows, which satisfied the traditional criterion that they had learned the procedure "meaningfully." But initially, their enumeration strategy was not based on a structuring that clearly identified exactly what was being counted. Indeed, although another pair of students made, unaided, the same discovery as Bethany and her partner, a third pair's explanation lacked a genuine appreciation of the difficulty. The students in this pair argued that since "a cube is in both the width and the length, it's okay to count it twice."

Encouraging the Development of Structural Reasoning

TO CULTIVATE THE DEVELOPMENT OF MEANingful enumeration strategies in geometry, instruction should encourage students to focus on spatial structuring (see, e.g., Battista and Berle-Carman [1995]). Problems should be presented in a way that allows students to construct their own personally meaningful solution strategies. Teachers can facilitate the construction of such strategies, not by "giving" solution procedures to students but by encouraging students to invent, reflect on, test, and

Fig. 8 Problem set 1

How many cubes fit in each box? Predict, then build the box and fill it with cubes to check.
Check your prediction for a box before going on to the next box.
Try to figure out a way to predict the number of cubes that would fit in any box.

	Pattern	Picture	Prediction	Actual
Box 1				
Box 2				
Box 3				
Box 4	The bottom of the box is six cubes long and five cubes wide. The box is four cubes high.			

publicly discuss strategies in a spirit of inquiry and problem solving.

Students who are learning to find the volume of rectangular boxes can start with problems like those in problem set 1 (**fig. 8**). Students first should be given several problems in which both the pattern and the box pictures are shown; then problems in which one picture is omitted; and finally, verbally stated problems.

Students can then move on to problems like those in problem set 2 (**fig. 9**). Here the goal is to ensure that students actually structure the set of volume units rather than use a numerical procedure that they do not fully understand. Rotely applying a procedure, such as $L \times W \times H$, will not work here. Neither will applying division when a student might reason, for example, that because seventy-two cubes fill the box and package A is made up of eight cubes, seventy-two cubes are divided by eight package A's. Although this strategy works for packages B and C, it does not work for package A because copies of this package do not completely fill the box, since we cannot break packages apart. Problem set 2 forces students to attend explicitly to how packages fit into boxes and to structure them properly for enumeration. A third problem set, which further encourages students to structure sets of packages that fit into boxes, can include such problems as those shown in **figures 3** and **5**.

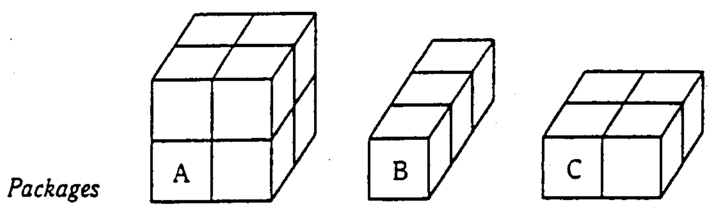
Students can go on to explore the use of standard units of volume, starting, for instance, with finding the number of cubic centimeters that fit into a closed, unmarked rectangular box. Because some students use actual centimeter cubes to solve the problem and others use rulers, students can have meaningful discussions about what linear measurements reveal about the structure of the configuration of cubes that fills the box.

Finally, students can be given problems in which the dimensions of boxes are not whole numbers. The following problem encourages students to perform the spatial structuring necessary to make sense of these situations:

> How many one-inch clay cubes will completely fill a box that measures 6 inches by 4 inches by 2 1/2 inches? The cubes can be cut apart.

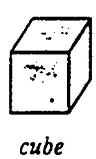
One viable way for students to structure the cubes is as two 6-by-4 layers of whole cubes, plus a 6-by-4 layer of half cubes. See **figure 10**.

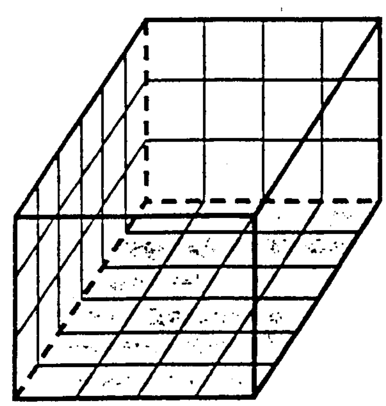
Fig. 9 Problem set 2



- How many of each package do you predict will fit in the box below? Use only one type of package at a time. You may not break packages apart.
- Predict, then make the box, and fill it with cube packages to check your predictions.

The bottom of the box is six cubes long and four cubes wide. The box is three cubes high.





The Importance of Predictions

FOR ALL PROBLEM SITUATIONS, STUDENTS should first make predictions, then check their predictions by using such concrete materials as cubes and paper boxes. Predicting first is essential because students' predictions are based on their current ways of structuring the cube or package arrays. Making and testing predictions encourage students to reflect on and refine their structuring, which helps them develop more powerful ways of conceptualizing and solving volume and packing problems. Having students merely make boxes and fill them with cubes does not promote nearly as much productive student reflection because *(a)* discrepancies between predicted and actual answers that stimulate reorganization of thought are greatly reduced and *(b)* students' attention is focused on physical activity rather than on their own thinking.

Student-Generated Procedures versus Formulas

EXAMPLES GIVEN IN THIS ARTICLE, ALONG WITH research, suggest that we reconsider the practice of teaching students to determine the number of cubes in a three-dimensional array by multiplying its length, width, and height—a practice found in mathematics textbooks for students in grades as low as grade 3. The research suggests that only about 30 percent of fifth graders consistently mentally construct a layer structure for three-dimensional arrays, a construction that is absolutely necessary, but certainly not sufficient, for understanding such a procedure (Battista and Clements 1996).

But even for these students, it is questionable whether the procedure of multiplying the dimensions describes accurately the personally meaningful ways they have developed for finding the number of cubes in a box. Indeed, when students are given instructional activities as suggested in this article, almost all of them construct enumeration procedures having two distinct steps: first, determine the number of cubes in a layer, which students may or may not accomplish with multiplication; second, account for the number of layers, usually with multiplication, repeated addition, or skip counting. Even students who use multiplication for both steps do not usually describe their procedure by using a three-factor product. For instance, two fifth graders wrote and described their general procedure for determining the number of cubes in a rectangular box as follows: They labeled the length as *A;* the width, *B;* and the height, *D* and said, "*A* times *B* equals *C*, that's the number on the bottom. *C* times *D* gives the total."

Thus, it is best to let students retain the procedure that almost all of them will develop if given appropriate instructional tasks: Find out how many cubes are in a layer, then multiply by the number of layers. This student-generated procedure not only is personally meaningful to students but is much more powerful than the traditional formula because it generalizes to all prisms and can even form part of the conceptual basis for integral calculus.

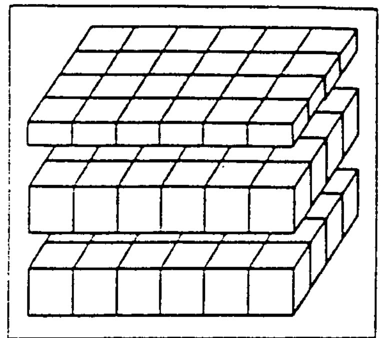


Fig. 10 Extending thinking to fractional dimensions

Conclusion

THE EXAMPLES DESCRIBED IN THIS ARTICLE indicate that students frequently determine the number of cubes or packages that fill a box without paying sufficient attention to the process of spatial structuring. They employ numerical operations that are insufficiently tied to a spatial structuring that can make these operations truly meaningful because most elementary and middle school students are unaware of the crucial importance that spatial structuring plays in dealing with geometry problems. We should not wonder that so many students made structuring errors in the package problems in figures 3 and 5—for many, it did not occur to them to look carefully at structuring the situation; for others, the opportunities for structuring arrays had been so infrequent that they had not yet become competent with this important process. Thus, if we want to increase the power of students' geometric reasoning, we must give them numerous opportunities to develop their spatial-structuring skills.

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