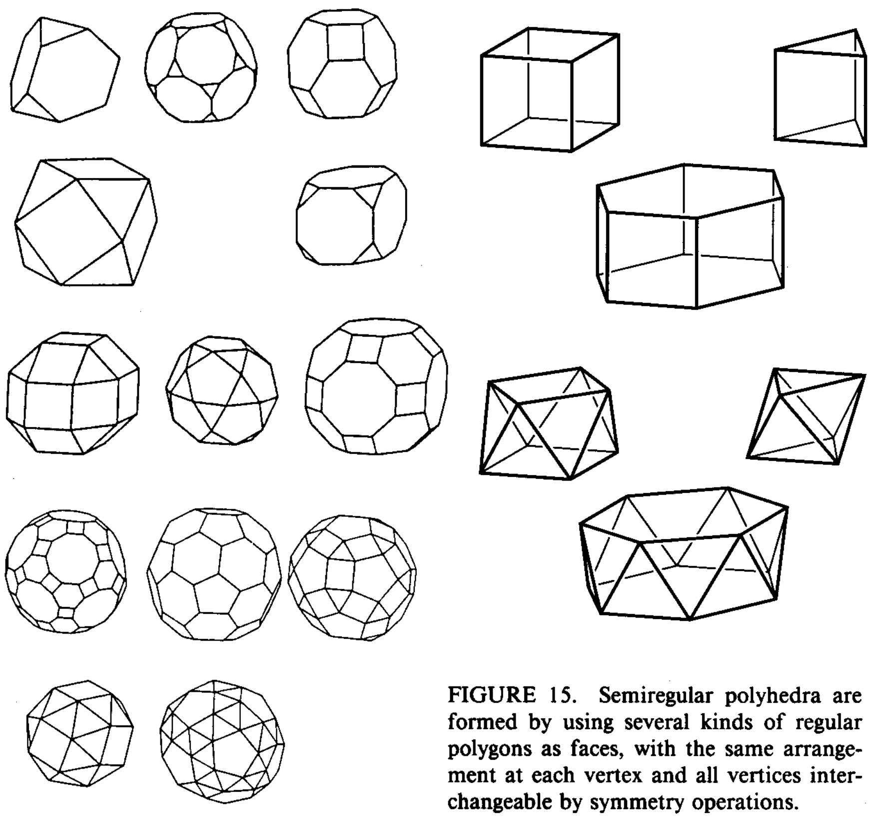

FIGURE 15. Semiregular polyhedra are formed by using several kinds of regular polygons as faces, with the same arrangement at each vertex and all vertices interchangeable by symmetry operations.

the arrangements in crystals will be orderly? Although these are profound and largely unsolved problems, a good working answer was given over thirty years ago by James Watson and Francis Crick in describing their discovery of the structure of DNA:[22]

> Wherever, on the molecular level, a structure of a definite size and shape has to be built up from smaller units ... the packing arrangements are likely to be repeated again and again and hence sub-units are likely to be related by symmetry elements.

In other words, nature builds modular structures that organize themselves according to certain rules. Repetition of the rules tends to lead to arrangements of modules that we call symmetrical.

Polyhedra provide a wealth of excellent examples of arrangements that are repeated again and again. When you build a cube with cardboard squares by attaching three squares to each corner, you are constructing a shape that satisfies a certain packing arrangement: it must be made of congruent regular polygons, and it must have the same number

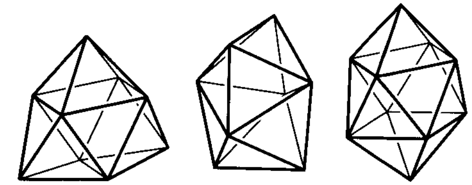

FIGURE 16. Convex deltahedra are formed from equilateral triangles arranged with differing types of vertex arrangements: three, four, or five triangles may be joined at a vertex.

at each corner. By generalizing this construction to other polygons, we obtain the five regular polyhedra (Figure 1). The arrangements can be further generalized to include the semiregular polyhedra (Figure 15), in which more than one kind of regular polygon can be used, and the convex deltahedra (Figure 16), all of whose faces are equilateral triangles but whose vertex arrangements need not all be the same.[17]

The cover design for the biological journal *Virology* contains an icosahedron. The story of the discovery of icosahedral symmetry in viruses and the ongoing efforts of scientists to link that symmetry to their subunit structures is very instructive.[17] Viruses are tiny capsules that contain an infective agent. The capsule is composed of protein subunits that group together to form a closed shell. Watson and Crick realized, in the course of early X-ray investigations into virus structure, that the shells of many viruses had polyhedral or helical forms. Subsequent studies showed that the polyhedra were often icosahedra, and this suggested many attractive models for the arrangement of the protein subunits. But more recently these models have been found to be incorrect. The connection between packing arrangements and overall symmetry in viruses remains an unsolved problem. Problems such as these lead also to new developments in mathematics: they force mathematicians to rethink their definitions and to broaden the scope of their investigations.

Lattices

From earliest times the beautiful shapes that we call crystals have been a source of wonder and admiration. Why do they have polyhedral forms when most other natural structures do not? Quartz crystals were the first to be studied; at first they were thought to be pieces of permanently frozen ice. (It is instructive that our word "crystal" comes from the Greek word *κρισταλλος*, which means ice.) By the seventeenth century, scientists began to suspect that the shapes of crystals reflected an orderly, patterned, internal structure. Long before the development of modern atomic theory it was suggested that crystals are made of stacks of tiny spheres that represented the basic particles of the structure, whatever those might be. Later the particles were represented as tiny bricks (Figure 17). Sphere packings and bricks (not necessarily rectangular) are still important models for crystal structure.

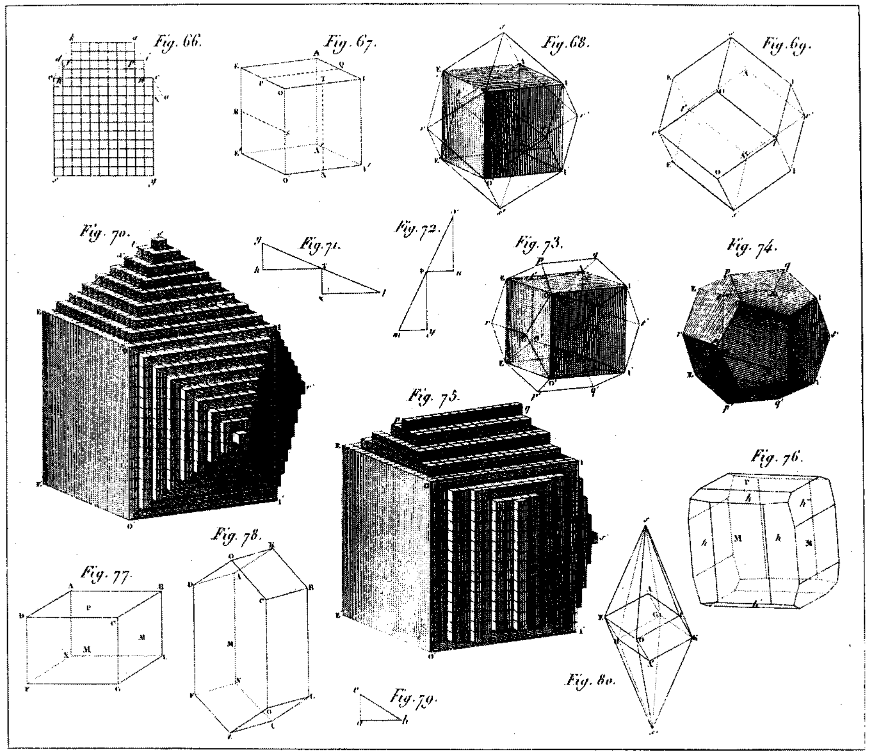

FIGURE 17. An 1822 concept of crystal structure in which various crystal shapes are imagined as being built from tiny rectangular bricks.

Whether we use spheres or bricks, the important idea is that of an orderly array. Let us explore this a little further. A one-dimensional *lattice* is a set of points equally spaced along a line. (Although we can draw only part of the set, we assume that it goes on forever.) All one-dimensional lattices are essentially alike, differing only in the spacing between points. But there are two basic kinds of two-dimensional lattices: one in which the points of the rows lie directly above one another, the other in which they are shifted horizontally (see Figure 18). Each point of a lattice "occupies" a certain portion of the plane, the region nearer to it than to any of the other lattice points. These regions, called Dirichlet domains, display the symmetry of the lattice in a corresponding brick model. The Dirichlet domains in two dimensions—the bricks—are always quadrilaterals or hexagons; within each lattice the regions about each of the points are congruent.

Lattices describe the underlying symmetries of patterns. Draw a one-dimensional lattice on two or three different sheets of tracing paper, and

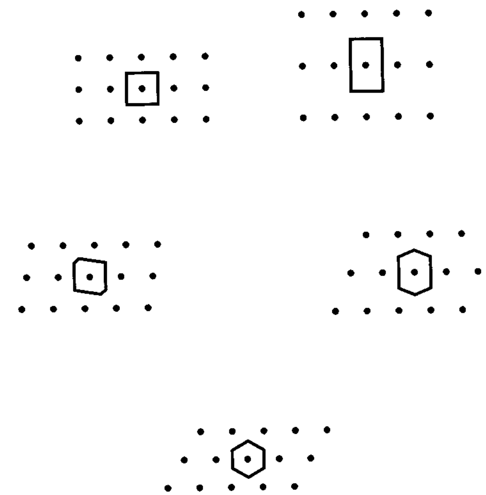

FIGURE 18. The symmetry of two-dimensional lattices is displayed by their Dirichlet domains, polygons centered at each lattice point which enclose the region of the plane closer to the enclosed lattice point than to any other. These polygons may be quadrilateral or hexagonal; for a given lattice they are all congruent.

use them to create two-dimensional lattices. You will quickly discover that you can change the symmetry of the lattice by shifting the relative positions of the rows: you can check the symmetry by recalculating the Dirichlet domains. No matter what you do, the symmetry will always be of one of the five types shown in Figure 18. It is an important fact that every two-dimensional repeating pattern, whether it is an arrangement of points or ellipses or polygons, a wallpaper pattern, or an Escher-like tiling of the plane, can be interpreted as a decoration of the Dirichlet domains associated with a lattice that belongs to one of these five symmetry types.

This simple observation raises a wealth of interesting questions. What kind of packing arrangements can we create if we replace the points by other shapes? What shapes can be fitted together without gaps to form orderly patterns? What do we mean by orderly? What are the possible ways to extend arrays to three dimensions? It turns out that there are only a small number of solutions to problems such as these, which explains why the same patterns reappear so often in crystal structures, trusses, biological tissues, honeycombs, wallpaper, textiles, and tiled floors.

Three-dimensional lattices have been used by mathematicians and scientists, beginning in the nineteenth century, to try to explain the arrangements of atoms in crystals. In three dimensions there are 14 symmetry types of lattices and 5 combinatorial types of Dirichlet domains (see Figure 19).

It is difficult to overestimate the importance of play with cubes and other blocks. Even one year olds enjoy building taller and taller towers and watching them fall down. Later, children use blocks to build