Geometric approaches to things

Geometric thinking is an absolute necessity in every branch of mathematics, and, throughout history, the geometric point of view has provided exactly the right insight for many investigations (complex analysis, for example).[6] Geometers (amateurs and professionals) seem to have a special stash of tricks of the trade.

Geometers use proportional reasoning. There is a whole family of geometers (who trace their ancestry back to Euclid) for whom a real number is a ratio of two magnitudes.[7] These are the people who delight in the beautiful theorems about proportions ("the altitude to the hypotenuse is the geometric mean between the segments into which it divides the hypotenuse," for example), who are somehow able to visualize the product and quotient of two lengths, and who begin a geometric investigation by looking for constant ratios.

Visualizing proportionality is hard. Computers might help students develop proportional reasoning in a variety of ways. Measure boxes that contain ratios can show how two lengths can change size but maintain the same ratio. Software that allows one to define dilations can help students estimate scale factors necessary to map one figure onto a similar one. Proportional reasoning is a necessary ingredient in vectorial methods and in the study of fractal geometry.

Proportions in geometry often express a beautiful blending of numerical and geometric languages. This is an example of a more general phenomenon.

Geometers use several languages at once. Except in high school texts, there are no treatments of geometry that use a single technique for solving problems. Among the languages used by geometers are local languages (turtle geometry, for example), vectors (including complex numbers), "analytic" geometry (coordinates), and algebraic languages (the language of algebraic number fields). And, these languages are often used in the *same* investigation. This multiplicity of languages points to the habit of using multiple of points of view.

Even though geometric investigations are carried out with several languages, geometric *results* always sound like geometry.

Geometers use one language for everything. For the past 150 years, the language of points, lines, angles, planes, surfaces, areas, and volumes has been applied to seemingly non-geometric phenomena, providing insight and coherence in many disparate branches of mathematics. For example, instead of saying that 1, 2, 3, and 4 are numbers that satisfy the equation $x + y + z + w = 10$, geometers (and most mathematicians), say that $(1, 2, 3, 4)$ is a "point" on the "graph" of $x + y + z + w = 10$. The entire graph of the equation (that is the collection of points that satisfy the equation) is called a "hyper-plane." Once we're calling things like $(1, 2, 3, 4)$ points, we might as well talk about vectors, and then we can define orthogonal vectors, and even the angle between two vectors.

The strategy is to take a familiar geometric idea, say the cosine of the angle between two vectors A and B, find a description of that idea that makes sense for the generalization (in this case, some algebraic expression[8] that gives the cosine), show that the description can be used as a *definition* of the idea for generalized "vectors" (in this case, you'd want the algebraic expression to always take values between -1 and 1, for example), and then to work with this new definition using familiar geometric language.

In one sense, this is a game, an example of extending the language. But it's more than a game: By finding a way to use the language of geometry to describe a new situation, we get a whole collection of insights that might be true in the new domain. So, in our geometry of points that look like $(1, 2, 3, 4)$, what's the proper analogy for a triangle? Do the angles of a triangle add up to 180°? Do two planes intersect in a "line?" Questions like these often point up fruitful lines of investigation. They also make geometry more powerful because they extend the domain over which geometric facts apply.

An example of using geometry-talk to gain new ways to look at things is in number theory. Around the turn of the century (this one), the mathematician Hensel was investigating ways for solving algebraic congruences modulo powers of primes. He invented a collection of techniques that he turned into a number system, the p-adic integers (p is a prime), and as the work progressed, it began to borrow heavily from the language of geometry. The geometry in the p-adic integers was strange indeed: every triangle was isosceles, and circles had infinitely many centers. But, once you get used to this strange land, geometric language gives you some ideas about what to expect, and it provides you with some interesting slants on arithmetic. As it turns out, the geometric analogies were more than just analogies: It's possible to realize the geometry of the p-adic integers as the geometry of very simple fractal-like subsets of the plane. So, things come full circle: The language of geometry is transported to a non-geometric situation as an aid to describing arithmetic phenomena. But then the language suggests that there might actually *be* an underlying geometry after all, and it turns out that the "non-geometric" situation has a concrete geometric model.

Now, all mathematicians appreciate the way that geometric language gives coherence to their discipline, but geometers seem to like another aspect of this approach: they love to use words like point and line when they are really talking about, say, numbers and sets, because they love the way everything hangs together.

Geometers love systems. Bill Kramer, a high-school mathematics teacher living in Dayton, Ohio, has been enjoying a hobby for about 20 years: Bill has defined a geometry that contains 25 points.[9] He has defined lines, triangles, measures, even rotations on his 25 points, and his hobby consists of seeing how far he can push the analogy with Euclidean geometry in this finite world. What attracts Bill to this work is the logical connectedness of it all; he asks what a reasonable definition for, say parallel lines would be, and then he sees if the classical theorems about parallels and, say, angles, hold up in his system.

Geometers like another kind of systematizing in which many special cases are combined into one large result. One way to do this is to look at *families* of geometric events.

Geometers worry about things that change. Because geometry was originally developed to describe two and three dimensional space, reasoning by continuity has always had an attraction for geometers. Continuity can be used to systematize disparate results. So, an angle formed by two chords has measure equal to half the sum of its arcs. Move the vertex of the angle towards the circle; one arc goes to 0 and the angle becomes an inscribed angle, and a new theorem is born. Then move the vertex outside the circle, to get another result, and finally, if you like, move it to infinity to see that parallel chords subtend equal arcs.

Sometimes, you expect things to change and they don't. Eventually, you learn how *useful* that is.

Geometers worry about things that don't change. Suppose you take a small rotation of, say, 2°, followed by a big vertical translation, say 80 feet straight up. What is the resulting map? A little experimenting suggests that it might be a rotation about some distant point. How could you check things further? One way would be to try to find the center of the alleged rotation. And one way to look for this center is to look for a point that *doesn't move* under the transformation.

This searching for invariants under transformations is a key ingredient in geometric investigations. For certain kinds of maps, this leads you into the theory of eigenvalues. For other kinds, you start thinking about topological invariants. Klein distinguished different *geometries* by the *theorems* that stayed true under the action of the respective transformation groups.

The habit of looking for invariants comes into play in another context: Invariance can be used to show that a given construction produces a well-defined function. The theorem about the "power of a point" is one of these: Define a function on $\mathbb{R}^2$ by drawing a line from a point P that intersects a circle O in two points A and B (A and B might be the same). Then the value of the function at P is defined as the product $PA \times PB$. The theorem is that this function is well-defined: It doesn't matter what line you draw through P.

One last thing about geometers.

Geometers love shapes. There is absolutely nothing to say here beyond what Marjorie Senechal says in her beautiful piece "Shape" in *On the Shoulders of Giants.*[11] In that article, Senechal breaks the study of shape into four broad categories. In addition to visualization, these include:

- Classification. Geometers classify shapes by congruence and similarity, by combinatorial properties (numbers of vertices or edges, for example), and by topological properties (number of "holes," for example).
- Analysis. Tools used to analyze shapes include symmetry (including self-similarity), regularity (tiling and packing properties), dissection, and combinatorics.
- Representation. Representations include models, drawings, computer graphics, maps, and projections.

Just look in a book or paper written by a geometer (Senechal or Coxeter, for example). There are pictures.

Cuoco, Al; Goldenberg, E. Paul; and Mark, June (December, 1996). Geometric Approaches to Things.
In the paper "Habits of Mind: An Organizing Principle for Mathematics Curriculum." *The Journal of Mathematical Behavior, 5*, (4), 375-402. Reproduced with permission from the publisher.