

variable. For instance, Fey and Good (1985) see the following as the key questions on which to base the study of algebra:

For a given function $f(x)$, find—

1. $f(x)$ for $x = a$;
2. x so that $f(x) = a$;
3. x so that maximum or minimum values of $f(x)$ occur;
4. the rate of change in f near $x = a$;
5. the average value of f over the interval (a, b) . (P. 48)

Under this conception, a variable is an *argument* (i.e., stands for a domain value of a function) or a *parameter* (i.e., stands for a number on which other numbers depend). Only in this conception do the notions of independent variable and dependent variable exist. Functions arise rather immediately, for we need to have a name for values that depend on the argument or parameter x . Function notation (as in $f(x) = 3x + 5$) is a new idea when students first see it: $f(x) = 3x + 5$ looks and feels different from $y = 3x + 5$. (In this regard, one reason $y = f(x)$ may confuse students is because the function f , rather than the argument x , has become the parameter. Indeed, the use of $f(x)$ to name a function, as Fey and Good do in the quote above, is seen by some educators as contributing to that confusion.)

That variables as arguments differ from variables as unknowns is further evidenced by the following question:

Find an equation for the line through $(6, 2)$ with slope 11.

The usual solution combines all the uses of variables discussed so far, perhaps explaining why some students have difficulty with it. Let us analyze the usual solution. We begin by noting that points on a line are related by an equation of the form

$$y = mx + b.$$

This is both a pattern among variables and a formula. In our minds it is a function with domain variable x and range variable y , but to students it is not clear which of m , x , or b is the argument. As a pattern it is easy to understand, but in the context of this problem, some things are unknown. All the letters look like unknowns (particularly the x and y , letters traditionally used for that purpose).

Now to the solution. Since we know m , we substitute for it:

$$y = 11x + b$$

Thus m is here a constant, not a parameter. Now we need to find b . Thus b has changed from parameter to unknown. But how to find b ? We use one pair of the many pairs in the relationship between x and y . That is, we select a value for the argument x for which we know y . Having to substitute a pair

of values for x and y can be done because $y = mx + b$ describes a general pattern among numbers. With substitution,

$$2 = 11 \cdot 6 + b,$$

and so $b = -64$. But we haven't found x and y even though we have values for them, because they were not unknowns. We have only found the unknown b , and we substitute in the appropriate equation to get the answer

$$y = 11x - 64.$$

Another way to make the distinction between the different uses of the variables in this problem is to engage quantifiers. We think: For all x and y , there exist m and b with $y = mx + b$. We are given the value that exists for m , so we find the value that exists for b by using one of the "for all x and y " pairs, and so on. Or we use the equivalent set language: We know the line is $\{(x,y): y = mx + b\}$ and we know m and try to find b . In the language of sets or quantifiers, x and y are known as *dummy variables* because any symbols could be used in their stead. It is rather hard to convince students and even some teachers that $\{x: 3x = 6\} = \{y: 3y = 6\}$, even though each set is $\{2\}$. Many people think that the function f with $f(x) = x + 1$ is not the same as the function g with the same domain as f and with $g(y) = y + 1$. Only when variables are used as arguments may they be considered as dummy variables; this special use tends to be not well understood by students.

Conception 4: Algebra as the study of structures

The study of algebra at the college level involves structures such as groups, rings, integral domains, fields, and vector spaces. It seems to bear little resemblance to the study of algebra at the high school level, although the fields of real numbers and complex numbers and the various rings of polynomials underlie the theory of algebra, and properties of integral domains and groups explain why certain equations can be solved and others not. Yet we recognize algebra as the study of structures by the properties we ascribe to operations on real numbers and polynomials. Consider the following problem:

$$\text{Factor } 3x^2 + 4ax - 132a^2.$$

The conception of variable represented here is not the same as any previously discussed. There is no function or relation; the variable is not an argument. There is no equation to be solved, so the variable is not acting as an unknown. There is no arithmetic pattern to generalize.

The answer to the factoring question is $(3x + 22a)(x - 6a)$. The answer could be checked by substituting values for x and a in the given polynomial and in the factored answer, but this is almost never done. If factoring were checked that way, there would be a wisp of an argument that here we are

generalizing arithmetic. But in fact, the student is usually asked to check by multiplying the binomials, exactly the same procedure that the student has employed to get the answer in the first place. It is silly to check by repeating the process used to get the answer in the first place, but in this kind of problem students tend to treat the variables as marks on paper, without numbers as a referent. In the conception of algebra as the study of structures, the variable is little more than an arbitrary symbol.

There is a subtle quandary here. We want students to have the referents (usually real numbers) for variables in mind as they use the variables. But we also want students to be able to operate on the variables without always having to go to the level of the referent. For instance, when we ask students to derive a trigonometric identity such as $2\sin^2 x - 1 = \sin^4 x - \cos^4 x$, we do not want the student to think of the sine or cosine of a specific number or even to think of the sine or cosine functions, and we are not interested in ratios in triangles. We merely want to manipulate $\sin x$ and $\cos x$ into a different form using properties that are just as abstract as the identity we wish to derive.

In these kinds of problems, faith is placed in properties of the variables, in relationships between x 's and y 's and n 's, be they addends, factors, bases, or exponents. The variable has become an arbitrary object in a structure related by certain properties. It is the view of variable found in abstract algebra.

Much criticism has been leveled against the practice by which "symbol pushing" dominates early experiences with algebra. We call it "blind" manipulation when we criticize; "automatic" skills when we praise. Ultimately everyone desires that students have enough facility with algebraic symbols to deal with the appropriate skills abstractly. The key question is, What constitutes "enough facility"?

It is ironic that the two manifestations of this use of variable—theory and manipulation—are often viewed as opposite camps in the setting of policy toward the algebra curriculum, those who favor manipulation on one side, those who favor theory on the other. They come from the same view of variable.

VARIABLES IN COMPUTER SCIENCE

Algebra has a slightly different cast in computer science from what it has in mathematics. There is often a different syntax. Whereas in ordinary algebra, $x = x + 2$ suggests an equation with no solution, in BASIC the same sentence conveys the replacement of a particular storage location in a computer by a number two greater. This use of variable has been identified by Davis, Jockusch, and McKnight (1978, p. 33):

Computers give us another view of the basic mathematical concept of variable. From a computer point of view, the name of a variable can be thought of as the address of some specific memory register, and the value of the variable can be thought of as the contents of this memory register.

In computer science, variables are often identified strings of letters and numbers. This conveys a different feel and is the natural result of a different setting for variable. Computer applications tend to involve large numbers of variables that may stand for many different kinds of objects. Also, computers are programmed to manipulate the variables, so we do not have to abbreviate them for the purpose of easing the task of blind manipulation.

In computer science the uses of variables cover all the uses we have described above for variables. There is still the generalizing of arithmetic. The study of algorithms is a study of procedures. In fact, there are typical algebra questions that lend themselves to algorithmic thinking:

Begin with a number. Add 3 to it. Multiply it by 2. Subtract 11 from the result. . . .

In programming, one learns to consider the variable as an argument far sooner than is customary in algebra. In order to set up arrays, for example, some sort of function notation is needed. And finally, because computers have been programmed to perform manipulations with symbols without any referents for them, computer science has become a vehicle through which many students learn about variables (Papert 1980). Ultimately, because of this influence, it is likely that students will learn the many uses of variables far earlier than they do today.

SUMMARY

The different conceptions of algebra are related to different uses of variables. Here is an oversimplified summary of those relationships:

<i>Conception of algebra</i>	<i>Use of variables</i>
Generalized arithmetic	Pattern generalizers (translate, generalize)
Means to solve certain problems	Unknowns, constants (solve, simplify)
Study of relationships	Arguments, parameters (relate, graph)
Structure	Arbitrary marks on paper (manipulate, justify)

Earlier in this article, two issues concerning instruction in algebra were mentioned. Given the discussion above, it is now possible to interpret these issues as a question of the relative importance to be given at various levels of study to the various conceptions.

For example, consider the question of paper-and-pencil manipulative skills. In the past, one had to have such skills in order to solve problems and in order to study functions and other relations. Today, with computers able to simplify expressions, solve sentences, and graph functions, what to do with manipulative skills becomes a question of the importance of algebra as a structure, as the study of arbitrary marks on paper, as the study of arbitrary relationships among symbols. The prevailing view today seems to be that this should not be the major criterion (and certainly not the only criterion) by which algebra content is determined.

Consider the question of the role of function ideas in the study of algebra. It is again a question of the relative importance of the view of algebra as the study of relationships among quantities, in which the predominant manifestation of variable is as argument, compared to the other roles of algebra: as generalized arithmetic or as providing a means to solve problems.

Thus some of the important issues in the teaching and learning of algebra can be crystallized by casting them in the framework of conceptions of algebra and uses of variables, conceptions that have been changed by the explosion in the uses of mathematics and by the omnipresence of computers. No longer is it worthwhile to categorize algebra solely as generalized arithmetic, for it is much more than that. Algebra remains a vehicle for solving certain problems but it is more than that as well. It provides the means by which to describe and analyze relationships. And it is the key to the characterization and understanding of mathematical structures. Given these assets and the increased mathematization of society, it is no surprise that algebra is today the key area of study in secondary school mathematics and that this preeminence is likely to be with us for a long time.

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CAN YOUR ALGEBRA CLASS SOLVE THIS?

Problem 3. Find all real values of x that satisfy

$$(x^2 - 5x + 5)^{x^2 - 9x + 20} = 1.$$

Solution on page 248

CAN YOUR ALGEBRA CLASS SOLVE THIS?

Problem 4. If p pencils cost c cents, how many pencils can be purchased for d dollars?

Solution on page 248