Session 10

Classroom Case Studies, Grades 6-8

This is the final session of the ***Patterns, Functions, and Algebra*** course! In this session, we will examine how the types of mathematical tasks involving algebraic thinking from the previous nine sessions might look when applied to students in your own classrooms. This session is customized for three grade levels. Select the grade level most relevant to your teaching.

The session for grades 6-8 begins below. Go to page 251 for grades K-2 and page 273 for grades 3-5.

Key Terms for This Session

Previously Introduced

- mathematical thinking tools [Session 1]
- algebraic ideas [Session 1]
- representation [Session 1]
- variable [Session 2]
- recursive description [Session 2]
- output [Session 2]]
- input [Session 2]
- closed-form description [Session 2]
- function [Session 3]
- backtracking [Session 6]

Introduction and Review

In the previous sessions, we explored the components of algebraic thinking: fundamental algebraic ideas (content) and mathematical thinking tools (process). You were asked to put yourself in the position of a mathematics learner, both to analyze your individual approach to solving problems and to get some insights into your own conception of algebraic thinking. It may have been difficult for you to separate yourself as a mathematics learner from yourself as a mathematics teacher. Not surprisingly, this is often the case! In this session, however, we will shift the focus to your classroom and to the approaches your students might take to mathematical tasks involving algebraic thinking. **[SEE NOTE 1]**

Learning Objectives

In this session, we will focus on your experiences as a classroom teacher, as you:

- Explore how algebraic thinking is developed at your grade level
- Examine problems for their algebraic content
- Analyze mathematical tasks and their connection to mathematical themes in the course
- Critique lessons at your grade level for algebraic thinking

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NOTE 1. This session focuses on developing algebraic thinking in the grades 6-8 classroom. We'll consider the mathematics content of the previous sessions and its relation to the mathematics you teach in your own classrooms. We'll explore how algebraic thinking develops at your grade level by analyzing mathematical tasks appropriate for the grades 6-8 classroom. We'll also look at lessons from existing curriculum materials and critique them in relation to how students are asked to demonstrate their thinking and in relation to how the mathematics reveals algebraic thinking.

NOTE 1 cont'd. next page

Part A: Classroom Video (30 MINUTES)

To begin the exploration of what algebraic thinking looks like in a classroom at your grade level, watch a video segment of a teacher who has taken the *Patterns, Functions, and Algebra* course and has adapted the mathematics to her own teaching situation. When viewing the video, keep the following three questions in mind: **[SEE NOTE 2]**

a. What fundamental algebraic ideas (content) is the teacher trying to teach? Think back to the big ideas of the previous sessions: patterns, functions, linearity, proportional reasoning, nonlinear functions, and algebraic structure.

b. What mathematical thinking tools (process) does the teacher expect students to demonstrate? Think back to the processes you identified in the first seminar: problem solving skills, representation skills, and reasoning skills.

c. How do students demonstrate their knowledge of the intended content? What does the teacher do to elicit student thinking?

VIDEO SEGMENT (approximate times: 17:58-22:33): You can find this segment on the session video approximately 17 minutes and 58 seconds after the Annenberg/CPB logo. Zero the counter on your VCR clock when you see the Annenberg/CPB logo.

In this video segment, Lolita Mattos introduces her students to the process of backtracking. She begins by giving her students an algorithm. She then asks them to undo the algorithm by reversing operations.

Problem A1. Reflect on questions (a), (b), and (c) above. **[SEE NOTE 3]**

NOTE 1, CONT'D.

Review

Begin the session by reviewing the mathematics content in the previous nine sessions: patterns, functions, linearity, proportional reasoning, nonlinear functions, and algebraic structure. You may want to think about one big idea in each of these topics.

Homework Review

Groups: Discuss any questions about the homework.

NOTE 2. Before examining specific problems at your grade level with an eye toward algebraic thinking, we'll watch another teacher—one who has also taken the course—teaching in her classroom. The purpose is not to be critical of the teacher's methods or teaching style. Instead, look closely at how the teacher brings out algebraic ideas in teaching the topic at hand, as well as how the teacher extends the lesson and asks questions that elicit algebraic thinking.

Review the meaning of algebraic ideas (the content of algebra) and mathematical thinking tools (the processes used in analyzing problems). Keep in mind questions (a), (b), and (c) as you watch the video.

NOTE 3. Groups: Work in small groups on Problems A1-A4. Share answers to Problem A3 especially, since backtracking is not a common method of solving equations.

Part A, cont'd.

Problem A2. How does Ms. Mattos incorporate the concept of doing and undoing into the solving of equations?

Problem A3. How do students think about operations as a result of backtracking?

Problem A4. How do students think about solving equations when using backtracking?

Part B: An Example for Developing Algebraic Thinking (25 MINUTES)

The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (2000) identifies algebra as a strand for grades Pre-K-12. The *Standards* identify the following concepts that all students should cover and comprehend: **[SEE NOTE 4]**

- Understand patterns, relationships, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Use mathematical models to represent and understand quantitative relationships
- Analyze change in various contexts

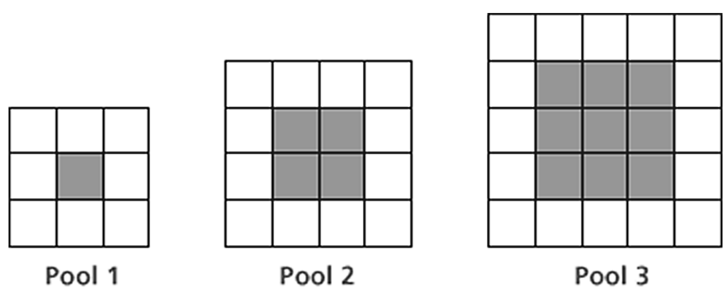
For the classroom in grades 6-8, understanding patterns includes the following expectations:

- Represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules
- Relate and compare different forms of representation for relationships
- Model and solve contextualized problems using various representations, such as graphs, tables, and equations

In this part, we'll look at problems that foster algebraic thinking as it relates to these standards, and explore ways of asking questions that elicit algebraic thinking. The situations we'll be exploring are representative of the kinds of problems you would find in some existing texts; in fact, you may recognize some of them! The goal is for you to examine these problems with the critical eye of someone who has taken this course and is beginning to view algebraic thinking with a different perspective.

Consider the situation below, appropriate for exploration in a grade 6-8 classroom:

Tat Ming is designing square swimming pools. Each pool has a square center that is the area of the water. Tat Ming uses blue tiles to represent the water. Around each pool there is a border of white tiles. Here are pictures of the three smallest square pools that he can design, with blue tiles for the interior and white tiles for the border. **[SEE NOTE 5]**



NOTE 4. Look at NCTM's recommendations for content in the algebra strand in the *Standards*, then look at the problem for designing square swimming pools. After reading the problem, you should work on Problems B1-B5.

NOTE 5. Read the commentary on the swimming pool problem in "Experiences With Patterning," by Joan Ferrini-Mundy, Glenda Lappan, and Elizabeth Phillips, in *Teaching Children Mathematics* (February 1997), pp. 282-288. The article can be found on the course Web site. Go to www.learner.org/learningmath and find *Patterns, Functions, and Algebra* Session 10, Grades 6-8, Part B, Note 5.

Now look at the problem on patterns from *IMPACT Mathematics, Course 1*. Think about asking questions that get at students' ability to think recursively. For example, if you knew there were 301 toothpicks for Term 100, could you state how many toothpicks were in the 102nd term?

Part B, cont'd.

Problem B1. What questions would you, as a mathematics learner, want to ask about this situation?

Problem B2. How do your questions reflect the algebra content in the situation?

Now focus on the questions you want the students in your classroom to consider. You may want to consider new ways to represent the relationships between the number of tiles of each color and the number of the square pools, and then use those representations to predict what will happen when the pools are very large.

Problem B3. What patterns, conjectures, and questions will your students find as they work with this situation?

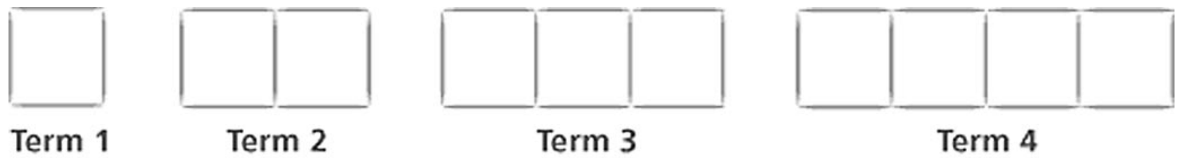
Problem B4. What questions could you as the teacher pose to elicit and extend student thinking at your grade level?

Problem B5. Recall the framework you explored in Session 2 in looking at patterns: finding, describing, explaining, and using patterns to predict. Which of these skills will your students use in approaching this problem?

Problem B6. Read the article "Experiences With Patterning" from *Teaching Children Mathematics,* which can be found on the course Web site. Go to www.learner.org/learningmath and find *Patterns, Functions, and Algebra* Session 10, Grades 6-8, Part B, Problem B6. What ideas mentioned seem appropriate for your classroom?

Part B, cont'd.

Problem B7. In this sequence there are 4 toothpicks in Term 1, 7 toothpicks in Term 2, and 10 toothpicks in Term 3.



How many toothpicks are in Term 4? If you continued the pattern, how many toothpicks would you need to make Term 5? Term 6? Term 10?

What questions could you ask to develop students' skills in describing this pattern?

Problem B8. What questions could you ask to develop students' skills in predicting?

The swimming pool problem is adapted from Algebra in the K-12 Curriculum: Dilemmas and Possibilities, *Final Report to the Board of Directors, by the NCTM Algebra Working Group (East Lansing, Mich.: Michigan State University, 1995).*

The pool problem and analysis of algebraic thinking are discussed in "Experiences With Patterning," by Joan Ferrini-Mundy, Glenda Lappan, and Elizabeth Phillips, in Teaching Children Mathematics *(February 1997), pp. 282-288.*

The pattern problem is taken from IMPACT Mathematics Course 1, *developed by Education Development Center, Inc. (New York: Glencoe/McGraw-Hill, 2000), p. 32.*

Part C: Patterns That Illustrate Algebraic Thinking (30 MINUTES)

Many students find that a context or situation helps them think about algebraic ideas. In the next problem, we'll look at different representations of a situation that lead to a solution. **[SEE NOTE 6]**

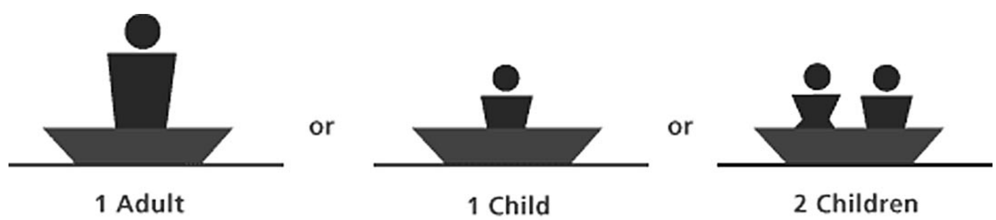
In this section we'll analyze problems at the 6-8 grade level for their algebraic content. For each problem, find a mathematical solution, then answer questions (a) through (d) listed below.

As you examine the problem below, consider these questions:

a. How would you solve the problem?
b. What is the algebraic content in the problem?
c. How do you think your students might solve the problem? What different representations might they use?
d. What question or questions might you ask to get at "doing and undoing"?

Here's the problem:

A group of 8 adults and 2 children needs to cross a river. They have a small boat that can hold either 1 adult, 1 child, or 2 children.



Problem C1. How many one-way trips does it take for the entire group of 8 adults and 2 children to cross the river? Tell how you found your answer.

Problem C2. How many trips in all for 6 adults and 2 children?

NOTE 6. Groups: Work on the crossing the river problem in small groups. Use concrete representations of the passengers, since "performing" the trips gives some insight into the solution of the problem.

Read "Patterns As Tools for Algebraic Reasoning," by Kristen Herbert and Rebecca Brown, in *Teaching Children Mathematics* (February 1997), focusing on different student approaches to solving the problem. This article can be found on the course Web site. Go to www.learner.org/learningmath and find *Patterns, Functions, and Algebra* Session 10, Grades 6-8, Part C, Note 6.

Part C, cont'd.

Problem C3. How many trips for 15 adults and 2 children?

Problem C4. How many trips for 23 adults and 2 children?

Problem C5. How many trips for 100 adults and 2 children?

Problem C6. Tell how you would find the number of one-way trips needed for any number of adults and 2 children to cross the river.

The river problem is taken from MathScape, *developed by Education Development Center, Inc. (New York: Glencoe/McGraw-Hill, 2000).*

Part D. More Problems That Illustrate Algebraic Thinking (30 MINUTES)

At the 6-8 grade level, there are many problems that prepare students for their later work in algebra. There is some evidence that more students are, in fact, taking a formal course in algebra in the eighth grade. The problems included in this session, however, may be used in courses both before and during students' first formal course in algebra. **[SEE NOTE 7]**

For each of the problems below, answer the following questions:

a. What algebraic content is in the problems?

b. What content does it prepare students for later?

c. How does this content relate to the mathematical ideas in this course?

d. How would your students approach this problem?

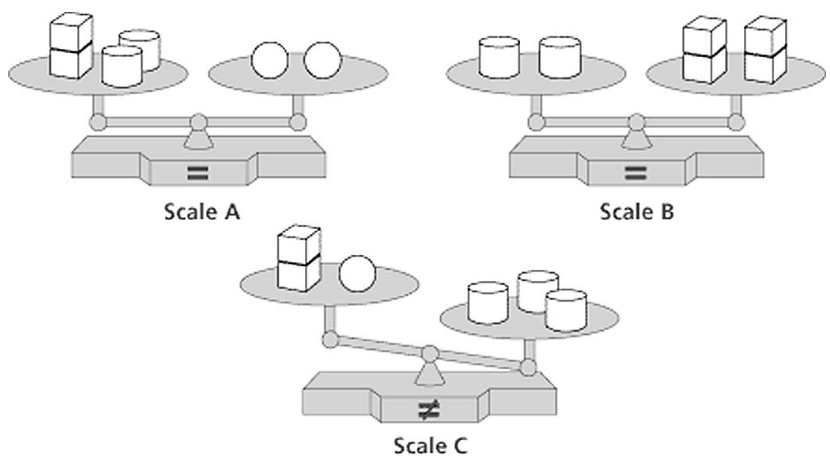
e. What are other questions that might extend students' thinking about the problem?

f. Does your current program in mathematics at your school include problems of this type?

Problem D1.

a. Which block, cylinder, sphere, or cube will balance Scale C?

b. List or draw the steps you followed to identify the block.

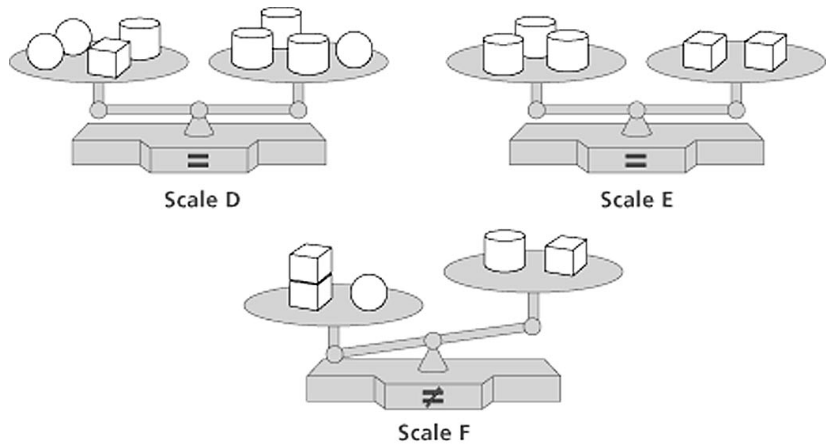


NOTE 7. The problems in this section, while preparing students for work in more formal algebra courses, demonstrate more informal approaches to algebraic concepts, with respect to their contexts and the models they employ. Work on Problems D1-D5, recording answers to questions (a) through (f) as you do each problem. It is important that you identify the mathematical content in each of the problems and how that content illustrates algebraic thinking.

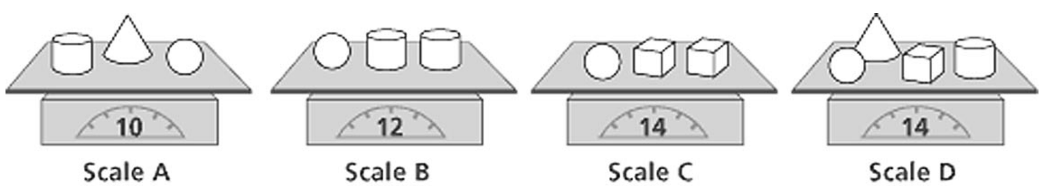
Part D, cont'd.

c. Which block, cylinder, sphere, or cube will balance scale F?

d. List or draw the steps you followed to identify the block.



Problem D2.



a. Find the weight of each block.

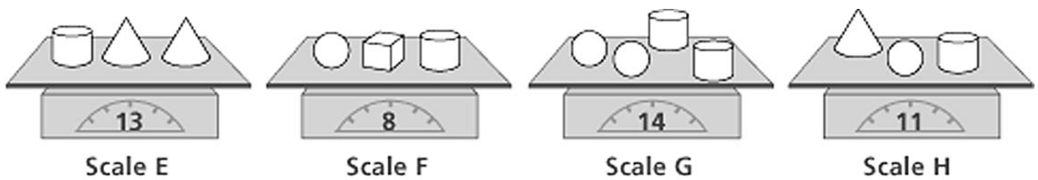
   cylinder = ________ pounds

   sphere = ________ pounds

   cube = ________ pounds

b. Write or draw the steps you followed to solve the problem.

Part D, cont'd.



c. Find the weight of each block.

cylinder = _______ pounds

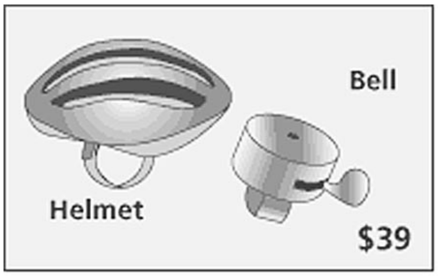
sphere = _______ pounds

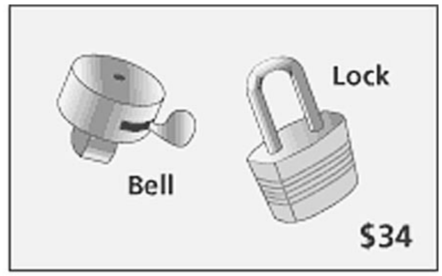
cube = _______ pounds

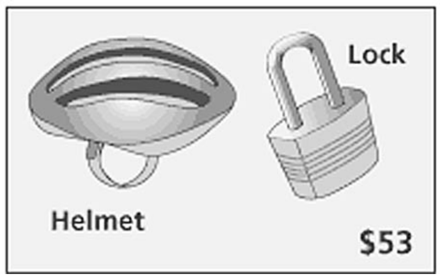
d. Write or draw the steps you followed to solve the problem.

Part D, cont'd.

Problem D3.







These signs tell about some items for sale. The same items have the same prices. Different items have different prices. How much is:

a. A helmet?

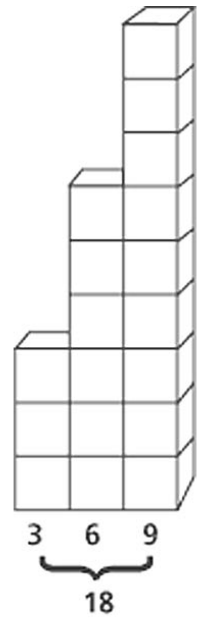
b. A bell?

c. A bicycle lock?

d. Explain how you figured out the prices.

Part D, cont'd.

Problem D4.



This is a staircase Deion built. He used 3 blocks for the 1st stair, 6 blocks for the 2nd stair, 9 blocks for the 3rd, and so on, using 3 more blocks for each higher stair.

a. Make a three-column table. In Column 1 show the stair number. In Column 2 show the number of blocks used. In Column 3 show the total number of blocks used to build the Stair 1 up to and including Stair 5. For example, for Stairs 1-3, Deion used a total of 3 + 6 + 9, or 18 blocks.

b. Let *S* represent the stair number. Let *N* represent the number of blocks. Complete this function rule to show the number of blocks needed to build stairs *S*.

   N = ______________

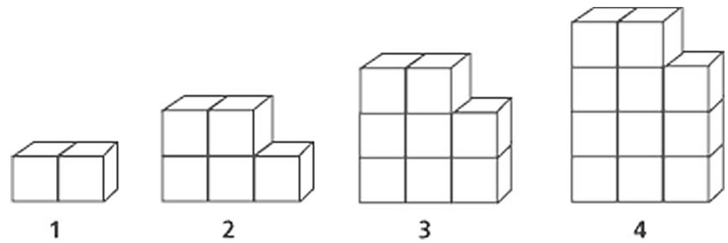
c. What is the number of blocks Deion would need to build Stair 10?

d. Explain your answer to the above question.

Part D, cont'd.

Problem D5.

This is a series of growing shapes. Imagine that the building pattern continues. How many blocks will it take to build:



a. Shape 8? ____

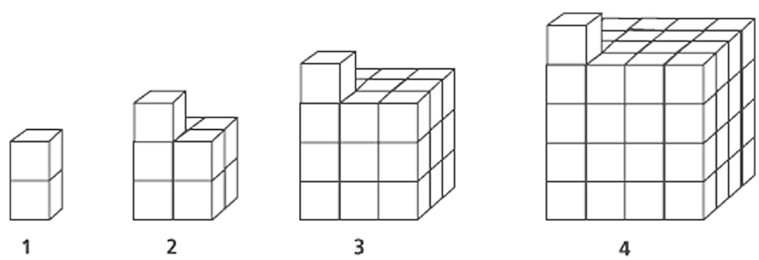
b. Shape 12? ____

c. Tell in words how you would build Shape 5.

d. Write a function rule to relate the number of blocks in a shape to the shape number. Let *N* represent the shape number. Let *B* represent the number of blocks in a shape.

   B = ____________

This is a series of growing cubes, with one small cube on top. Imagine that the building pattern continues. How many blocks will it take to build:



e. Cube 7? ____

f. Cube 20? ____

g. Tell in words how you would build Cube 5.

h. Write a function rule to relate the number of blocks in a cube to the cube number. Let *N* represent the cube number. Let *B* represent the number of blocks in a cube.

   B = ___________

Problems D1-D5 are taken from Groundworks: Algebraic Thinking, Grade 7, *by Carole Greenes and Carol Findell (New York: Creative Publications, Wright Group/McGraw-Hill 1998). The above material may not be reproduced without the written permission of Creative Publications.*

Part E: Critiquing Student Lessons
(20 MINUTES)

In the grades 6-8 curriculum, students are frequently asked to think about patterns, but often their "pattern sniffing" skills end with simply finding the next object. **[SEE NOTE 8]**

For each of the problems below, answer the following questions:

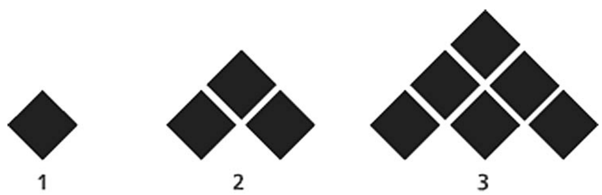
a. What algebraic ideas are in this problem?

b. How are patterns used in this problem?

c. What mathematics do you think students would learn from this problem?

d. How would you modify the problem, or what additional questions might you ask, to incorporate the framework for analyzing patterns?

Problem E1. How many tiles will be in the 8th figure of this pattern? How do you know?



NOTE 8. The two lessons in this section take a qualitatively different approach to students' thinking about patterns. The types of questions asked in the *Math in Context* patterns problem allow students to use the physical representation to make conjectures about the pattern's rule. In Problem E1, students are simply asked to supply a specific term in the sequence. In the framework developed on patterns, we were careful to include habits of mind that extend beyond finding a specific term. How do you know there are a specific number of dots or blocks in the sequence? If you know one term in the sequence, how could you get the next? Can you find a rule that will give you the number of dots or blocks for any term? These kinds of questions extend students' thinking about predicting with patterns.

Part E, cont'd.

Problem E2.

a. Make a drawing of the next V-pattern.
b. Is it possible for a V-pattern to have 84 dots? Why or why not?
c. How many pairs are there in each V-pattern?
d. How many dots will be in the 6th V-pattern?
e. Make a V-pattern with 19 dots.
f. Fill in the missing values in the table below. "V-Number" means the number of pairs. Describe any patterns you see.

V-Number	Number of Dots
1	3
2	5
3	7
4	?
5	?
6	?

Problem E1 is taken from Gateways to Algebra and Geometry, An Integrated Approach, *by John Benson, Sara Dodge, Walter Dodge, Charles Hamberg, George Milauskas, and Richard Rukin (Evanston, Ill.: McDougal Littell Inc., 1997). Problem E2 is taken from* Ups and Downs, Student Book, *part of the* Mathematics in Context *series (Chicago: Encyclopaedia Britannica, Inc., 1998), p. 6.*

Homework

Problem H1. Interview an algebra teacher. Pick one of the problems from the previous pages and ask the following:

a. How does the content of this problem prepare students for algebraic thinking in their class?

b. Why do they think this content is important?

c. How could this problem be extended for students in their class?

Problem H2. Look at a problem in your own mathematics program for your grade level that you think illustrates algebraic thinking. If you were to teach this problem after taking this course, how might you modify or extend it to bring out more of the content of algebraic thinking?

Solutions

Part A: Classroom Video

Problem A1. Reflection on the three questions should include the ideas described below.

a. The fundamental algebraic idea (content) in this video is the notion of inverse function as reversing processes and using inverse operations.

b. The teacher expects students to write a function rule to start, with a relationship shown as an equation, and then backtrack to find a rule for the inverse function.

c. Students demonstrate their knowledge of the intended content by drawing a flow chart for a multi-step function, drawing the backtracking flow chart, and then writing an inverse function.

Problem A2. Ms. Mattos asks students to focus on properties of operations, not properties of numbers.

Problem A3. The students discuss operations and inverse operations when they backtrack.

Problem A4. The students describe backtracking as reversing a process and using inverse operations. This process helps them decide what to do when they solve equations.

Part B: An Example for Developing Algebraic Thinking

Problem B1. Answers will vary. One question might ask how to use the pictures to help find the relationship between the pool number and the numbers of white and blue tiles.

Problem B2. Answers will vary. For the question above, you might discuss building each pool. For example, the blue part of Pool 1 is a 1-by-1 square and takes one blue tile, the blue part of Pool 2 is a 2-by-2 square and takes four blue tiles, etc.

Problem B3. Answers will vary. At this level, students should be able to write rules for the number of tiles of each color for each pool. The blue tiles are always in the shape of a square, with n^2 blue tiles needed for Pool n. The number of white tiles is always a multiple of four, with $4(n + 1)$ white tiles needed for Pool n.

Problem B4. Answers will vary. At this level, teachers should encourage students to describe the shape of the blue tiles and write a rule to represent the relationship between the pool number and the number of blue tiles. Then ask the students to think about putting white tiles around that blue pool. How many white tiles do you need for the corners? How many more for the bottom and top? Now think about putting tiles around the sides. How many white tiles does this require in all?

Problem B5. Answers will vary. At this level, students should actually build the pools with two different color square tiles and then describe what they built, thinking of the relationship between the pool number and the number of each color tile needed to build it. The process of building will help students put the patterns into words and symbols. Many students will use all these skills when solving this problem.

Problem B6. Answers will vary. All of the ideas described in the article's Grade 3-4 section are appropriate for this level student. Many students at this level will be able to answer questions posed in the article's Grade 5-6 section.

Problem B7. Answers will vary. Ask students to tell you how to build the 1st term. How many toothpicks will you need? Now describe how to build the 2nd term from the 1st. How many more toothpicks are needed? How many more toothpicks are needed to get from the 10th to the 11th term? Does this same rule work for getting from the 1st to the 2nd term?

Solutions, cont'd.

Problem B8. Answers will vary. Often students see a rule to get from one entry to the next. Teachers should help students think of ways to connect each entry to its place in the list. Questions like those shown above will help get the rule $N = 3n + 1$, where N is the number of toothpicks needed to build term n.

Part C: Patterns That Illustrate Algebraic Thinking

Possible responses for questions (a) through (d), which apply to all the problems in Part C, are as follows:

a. Answers will vary. Most people draw a diagram showing the number of trips to get 1 or 2 adults across the river and then generalize to 8 adults.

b. Answers will vary. The problem requires students to find a way to represent the problem, look for a pattern, and generalize. The problem can be thought of as a recursive pattern.

c. Answers will vary. Most students draw a diagram with arrows showing people crossing the river in each direction. This arrow representation allows students to "see" the number of trips required for each adult.

d. Answers will vary. Some possible questions include: What sequence of events must happen to get 1 adult across the river? Where are the 2 children at the end of this sequence? How many trips does the sequence require? What must happen so that 1 adult and 2 children are across the river? How does this change if there are 2 adults? Think of "undoing" your sequence to find out the number of adults with 2 children that required 13 trips to cross the river.

Problem C1. The answer is 33 trips: 4 trips to bring over each adult plus 1 more trip for the 2 children.

Problem C2. The answer is 25 trips: 4 trips to bring over each adult plus 1 more trip for the 2 children.

Problem C3. The answer is 61 trips: 4 trips to bring over each adult plus 1 more trip for the 2 children.

Problem C4. The answer is 93 trips: 4 trips to bring over each adult plus 1 more trip for the 2 children.

Problem C5. The answer is 401 trips: 4 trips to bring over each adult plus 1 more trip for 2 two children.

Problem C6. The answer is $(4N + 1)$ trips: 4 trips to bring over each adult plus 1 more trip for the 2 children.

Part D. More Problems That Illustrate Algebraic Thinking

Problem D1. For the first set of scales the answer is 1 cube on the left pan. For the second set of scales the answer is 1 cylinder on the right pan

a. The mathematical content is balance, the notion that the objects on the lower pan on a pan balance weigh more than the objects on the higher pan, and that one object that balances with two or more objects is the heaviest. This prepares students for understanding equality as balance and the notion that adding or removing the same items from both sides of a balance retains balance.

b. This content introduces equality as balance, setting the stage for solving equations by manipulating them while maintaining balance.

c. This content illustrates methods for solving linear equations.

d. Students approach these problems in several ways:

   - From Scale B, 2 cubes balance 1 cylinder.

Solutions, cont'd.

- Replace the 2 cylinders on Scale A with 4 cubes.
- Then 2 spheres balance 6 cubes.
- That means 3 cubes balance 1 sphere.
- Replace the sphere on Scale C with 3 cubes. That gives 5 cubes on the left.
- The 3 cylinders on the right need 6 cubes. To balance the scale, add 1 more cube.

e. To extend students' understanding, ask questions such as the following:

- Look at Scale A. Which block weighs the most? How do you know?
- Look at Scale B. Which weighs more, a cylinder or a cube?
- How many cubes balance 1 cylinder? How do you know?

f. Answers will vary. Very few textbooks contain problems of this type.

Problem D2. For the first set of scales the answers are: cylinder = 3, cube = 4, sphere = 6, cone = 1. For the second set of scales the answers are: cylinder = 5, cube = 1, sphere = 2, cone = 4.

a. The mathematical content is the concept of variable and solving four equations in four variables.

b. This content prepares students for understanding that they can replace a variable with its value.

c. This content illustrates methods for solving systems of linear equations.

d. Students approach these problems in several ways. One way is shown for each problem.

- For the first set of scales: Compare Scales A and D. Replace the cylinder, cone, and sphere on Scale D with 10 pounds. That means that the cube weighs 4 pounds (14 – 10). Replace the 2 cubes on Scale C with 8 pounds (2×4). That means the sphere weighs 6 pounds (14 – 8). Replace the sphere on Scale B with 6 pounds. Then the 2 cylinders weigh 6 pounds (12 – 6), and 1 cylinder weighs 3 pounds ($6 \div 2$). Then on Scale A, replace the cylinder and the cube with 9 pounds (6 + 3), to show the cone weighs 1 pound (10 – 9).

- For the second set of scales: From Scale G, if 2 spheres and 2 cylinders weigh 14 pounds, then 1 sphere and 1 cylinder weigh 7 pounds ($14 \div 2$). Replace 1 cylinder and 1 sphere on Scale H with 7 pounds to show that 1 cone weighs 4 pounds (11 – 7). Replace 1 cylinder and 1 sphere on Scale F with 7 pounds to show that 1 cube weighs 1 pound (8 – 7). Replace 2 cones on Scale E with 8 pounds (4 + 4) to show that 1 cylinder is 5 pounds (13 – 8). Since we know that 1 cylinder and 1 sphere weigh 7 pounds, then 1 sphere is 2 pounds (7 – 5).

e. To extend students' understanding, ask them to describe other ways to solve the problems and ask questions, such as the following:

- If each cylinder on Scale B weighs 5 pounds, how much does the sphere weigh? How do you know?
- If the sphere on Scale C weighs 4 pounds, how much does each cube weigh? How do you know?
- Compare Scales A and D. How much does the cube weigh? How do you know?

f. Answers will vary. Very few textbooks contain problems of this type.

Problem D3. The answers are: helmet = $29, bell = $10, lock = $24.

a. The mathematical content is the concept of variable and solving three equations in three variables.

b. This content prepares students for understanding that they can replace a variable with its value.

c. This content illustrates methods for solving systems of linear equations.

Solutions, cont'd.

d. Students approach these problems in several ways. Two ways are shown.

- Because the first two pictures each show a bell, and the 1st price is \$5 more than the 2nd, we know that the helmet costs \$5 more than the lock. In the 3rd picture, the helmet and lock together cost \$53. If the helmet was \$5 cheaper, the two would cost the same, and the combined price would be\$48 (53 – 5). Then each item would cost \$24 (48 ÷ 2). Thus, the lock costs \$24 and the helmet costs \$29 (24 + 5). Because the helmet and bell together cost \$39, the bell is \$10 (39 – 29).
- Adding all the pictures together shows that 2 helmets and 2 locks and 2 bells would cost 39 + 34 + 53, or \$126. Because 2 of each item costs \$126, then 1 of each item would cost \$63 (126 ÷ 2). The 1st picture shows that the helmet and bell cost \$39, so the lock must cost \$24 (63 – 39). Similarly, the helmet costs \$29 (63 – 34), and the bell costs \$10 (63 – 53).

e. To extend students' understanding, ask questions such as the following:

- How are the first two pictures the same?
- How are they different?
- What causes the difference?
- How does this difference help you solve the 3rd picture?

f. Answers will vary. Very few textbooks contain problems of this type.

Problem D4.

The answers are: 1, 3, 3; 2, 6, 9; 3, 9, 18; 4, 12, 30; 5, 15, 45. The rule for the number of blocks for stair N is $N = 3S$, so for Stair 10 it takes 3×10, or 30 blocks.

a. The mathematical content is function.

b. This content prepares students for making tables of values to represent data and for writing functions from those tables.

c. This content illustrates the representation of a function as a scenario, a table of values, and an equation.

d. Students approach these problems in several ways. One way is shown here.
Each stair requires 3 more blocks than the prior stair. The 1st stair requires 3 blocks, the 2nd stair requires 2×3 blocks, the 3rd stair requires 3×3 blocks, so the Sth stair will require 3 x S blocks.

e. To extend students' understanding, ask them to find the rule for the total number of blocks (T) needed to build S stairs. [That rule is $T = (3 / 2)S(S + 1)$.]

f. Answers will vary. Very few textbooks contain problems of this type.

Problem D5. For the first set of shapes the rule is $B = 2 + 3(N - 1) = 2N + (N - 1) = 3N - 1$. For the second set of shapes the rule is $B = 1 + N^3$.

a. The mathematical content is inductive reasoning and writing functions.

b. This content prepares students to identify, continue, and generalize patterns.

c. This content illustrates the difference between linear and nonlinear functions.

Solutions, cont'd.

d. Students approach these problems in several ways. One way is shown here.
   - For the first set of shapes: The 1st shape uses 2 blocks. The 2nd shape uses 2 blocks + 1 set of 3 blocks. The 3rd shape uses 2 blocks + 3 sets of 3 blocks. The *N*th shape will use 2 blocks + ($N - 1$) sets of 3 blocks.
   - For the second set of shapes: The 1st shape uses a $1 \times 1 \times 1$ cube of 1 block with 1 block on top. The 2nd shape uses a $2 \times 2 \times 2$ cube of 8 blocks with 1 on top. The 3rd shape uses a $3 \times 3 \times 3$ cube of 27 blocks with 1 on top. The *N*th shape uses an $N \times N \times N$ cube of N^3 blocks with 1 on top.

e. To extend students' understanding, ask questions such as the following:
   - How would you build the 5th or 6th or 10th shape?
   - How does telling how to build the shape help you write the function rule?
   - Describe how to build each shape in the first set of shapes so that you produce each of the different function rules shown in the answers.

f. Answers will vary. Very few textbooks contain problems of this type.

Part E: Critiquing Student Lessons

Problem E1.

a. This problem shows the first three figures of a growing pattern, and requires students to find the number of tiles needed to make the 8th figure in the pattern.

b. This problem uses a growing pattern, and the *n*th term uses *n* more tiles than the previous term.

c. This problem can be used to help students learn to write a general rule for the *n*th term of a function.

d. This problem could be extended to find the number of tiles needed for the 10th, 100th, and then the *n*th term. Ask students to describe how to build each term. Look for answers that relate each term to the previous term, and then relate the figure to its place in the pattern. For example, students might notice that the 1st term uses 1 tile, the 2nd term uses 1 + 2, or 3 tiles, the 3rd term uses 1 + 2 + 3, or 6 tiles, and so forth. The *n*th term uses $1 + 2 + 3 + ... + n$ tiles.

Problem E2.

a. This problem also shows the first three figures of a growing pattern. This problem not only requires students to find the number of dots needed to make the 6th figure in the pattern, however; it also requires students to determine whether it is possible for one of the terms to use 84 dots.

b. This problem uses a growing pattern, and the *n*th term uses 2 more dots than the previous term.

c. This problem can be used to help students learn to write a general rule for the *n*th term of a function.

d. This problem asks students to find the number of dots needed for the100th term, but could be further extended to ask for the number of dots needed for the *n*th term. Again ask students to describe how to build each term. Look for answers that relate each figure to its place in the pattern. For example, students might notice that the 1st term uses 1 dot + 1 pair of dots, the 2nd term uses 1 dot + 2 pairs of dots, the 3rd term uses 1 dot + 3 pairs of dots, and so forth. The *n*th term uses 1 dot + *n* pairs of dots, for a total of $2n + 1$ dots.