Session 6

Solving Equations

Key Terms for This Session

Previously Introduced

- axes [Session 1]
- function [Session 3]
- variable [Session 2]
- slope [Session 5]
- whole number [Session 3]

New in This Session

- set
- covering up
- integer
- solution set
- backtracking
- intercept
- equivalence
- intersection
- false position
- system of equations

Introduction and Review

In the previous session, we looked at linearity in different situations. We used spreadsheets to work with linear functions in tables, equations, and graphs. We also explored connections between rates and slopes, and examined the role of independent and dependent variables in linear functions. Now that we have explored characteristics of equations for lines, we will develop strategies for solving linear equations in this session. **[SEE NOTE 1]**

Learning Objectives

In this session we will explore different methods for solving equations. We will:

- Learn more about the meaning of the equal sign
- Explore the connection between equality and balance
- Solve equations by balancing, working backwards, and inverting operations
- Explore the strengths and limitations of different models for solving equations
- Solve systems of equations

NOTE 1. The problems in this session progress from informal to formal strategies for solving equations. One of the initial stumbling blocks to solving equations is making sense of the equal sign. In Part A, we will explore different interpretations of the equal sign, differentiating between the result of a process and an equivalence relation.

Later in the session, we will introduce historical and informal strategies for solving equations, such as the method of false position. In this method, we'll take a guess and then modify that guess so that it is a solution to the equation.

We'll also introduce the method of backtracking. This method involves solving equations by reasoning backwards from the answer, undoing the operations in reverse order. This is an informal method that seems intuitive to many students. It coincides with a view of equations primarily as a process. (Take a number, multiply it by 2, and add 1. The result is 12. What is the number?)

NOTE 1 cont'd. next page

Part A: Equality and Balance (30 MINUTES)

Equality

We have been using the equal sign throughout previous sessions. In this part we are going to look at the meaning of equality, which will give us some insight on using equality and its properties to solve equations. **[SEE NOTE 2]**

Problem A1. Examine the following equations. Each equation makes a statement about quantities. Is each statement true always, sometimes, or never? Think about your reasoning for each statement.

a. $5 + 3 = 8$

b. $2 + 14 = 12$

c. $5 + 3 = y$

d. $x + 3 = y$

e. $3x = 2x + x$

f. $3x = 3x + 1$

Problem A2. If someone asked you to solve the equations in Problem A1, what would the solution sets be? **[SEE TIP A2, PAGE 162]**

NOTE 1, CONT'D.

The traditional method of solving a linear equation by doing the same thing to both sides belies a more static interpretation of an equation. Picturing this method as a series of bags and blocks on a scale can help us to think of algebraic expressions as objects. As the equations get more complicated, however, the balance model becomes less appropriate, just as backtracking does not work with certain equations.

Review

Groups: Discuss any questions that came up on the homework. Be sure to look at solutions to the "undoing" problem (Problem H4, Session 5), because "undoing," or solving, equations will be the focus of today's session.

NOTE 2. The problems in this section use the balance scale as a model for the equality relationship. **Groups:** Discuss Problems A1-A3 as a whole group before going on to solve the balance scale puzzles.

For the equation in Problem A3, the equal sign acts as a symbol for students to do something or to perform the given operation. In other words, the equal sign connotes "give the answer to what comes before." This is a beginning notion of equality that at times interferes with a broader concept of equivalence. In fact, many students in elementary and middle school will argue that the equation in Problem A1(c) is impossible because "y" is not an "answer."

Although the equal sign signifies an equivalence relationship in all of the equations, the solution sets are different for some of the equations.

Part A, cont'd.

Problem A3. A chalkboard reads:

10 + 21 = ?

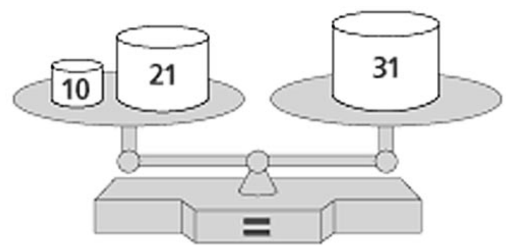
a. Replace "?" with something that makes the equation always true.

b. Replace "?" with something that makes the equation always false.

c. Replace "?" with something that makes the equation sometimes true and sometimes false.

Modeling With Balance Scales

A balance scale is a good visual model for representing an equivalence relationship. Picture a two-pan balance scale, with two weights on the left side and one on the right. If the weights on the left side are 10 and 21 grams, and the weight on the right side is 31 grams, the scale will be balanced. **[SEE NOTE 3]**



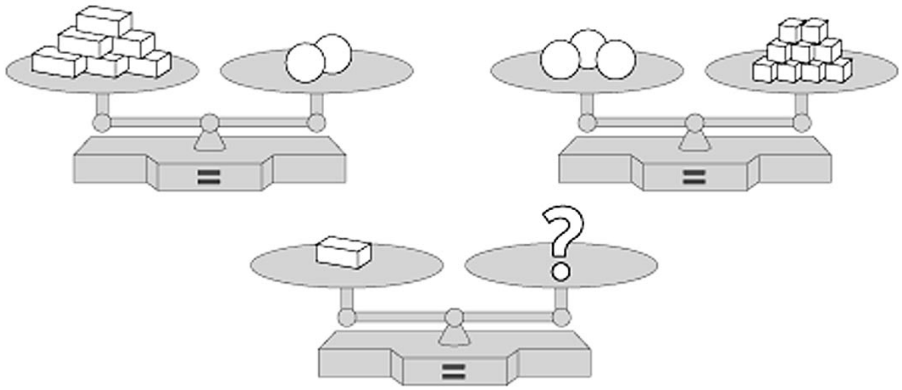
As you will see in Part C, using balance scales is also a good transition to more formal, symbolic techniques for solving equations.

NOTE 3. Groups: After discussing Problems A1-A3, work in pairs or small groups on the balance problems in Problem A4-A6.

Look for ways to think about replacing specific shapes for other shapes. The point here is not to use algebraic notation or equations to represent these situations. This sort of reasoning is essential in making the transition to solving equations, so it's an important step in the process. **Groups:** Share your solutions with the whole group, making sure to attend to the kind of reasoning used in solving Problems A4-A6.

Part A, cont'd.

Problem A4. In this balance puzzle, determine what will balance with the rectangle in the third scale. Throughout the three scales, the same shape always has the same weight.



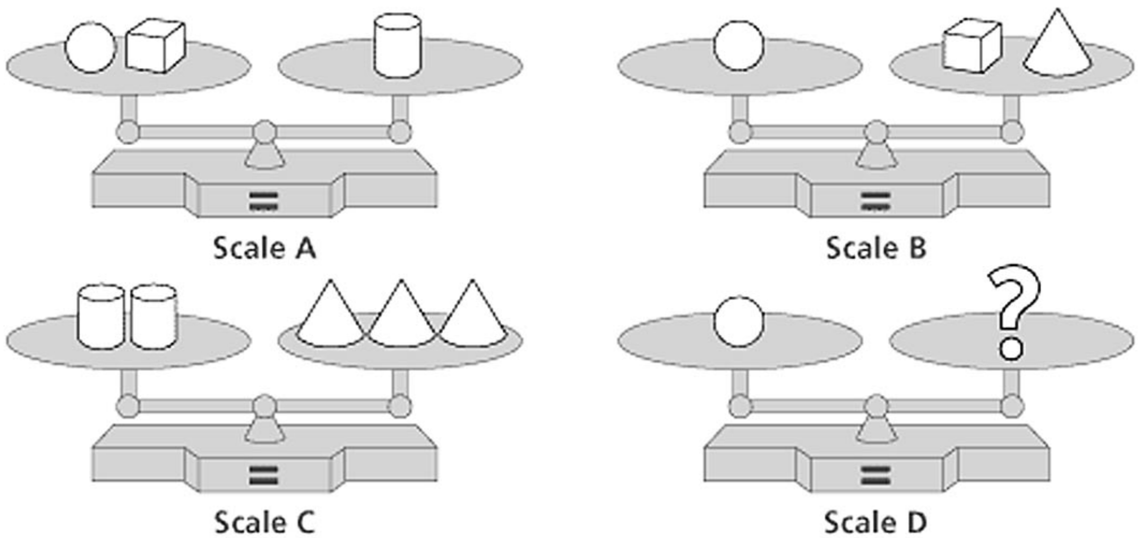
VIDEO SEGMENT (approximate times: 5:51-8:06): You can find this segment on the session video approximately 5 minutes and 51 seconds after the Annenberg/CPB logo. Zero the counter on your VCR clock when you see the Annenberg/CPB logo.

In this video segment, Frederick explains his solution to Problem A4. Watch the segment after you have completed Problem A4 and compare your strategy with Frederick's. If you get stuck on the problem, you can watch the video segment to help you.

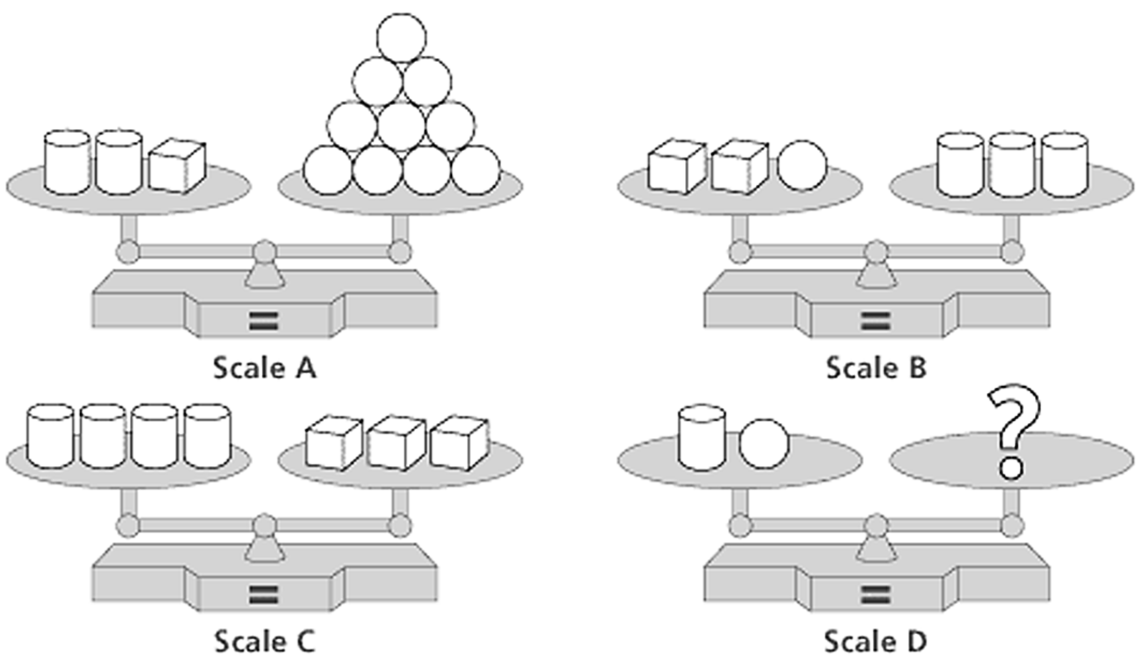
What strategies did you use to solve the problem? See if you can use Frederick's or your own strategy to solve the following problems.

Part A, cont'd.

Problem A5. What might be a solution for scale D, assuming that the same shapes have the same weight?
[SEE TIP A5, PAGE 162]



Problem A6. What might be a solution for scale D, assuming that the same shapes have the same weight? Note: The shapes in this problem may not be the same weight as the shapes in the previous problem.



Part A, cont'd.

Problem A7. Look at the following equations from Problem A1. For each equation, draw a balance scale to represent the equation. How can you use balance to decide when an equation is true or false?

a. $5 + 3 = 8$

b. $2 + 14 = 12$

c. $5 + 3 = y$

d. $x + 3 = y$

e. $3x = 2x + x$

f. $3x = 3x + 1$

Problems in Part A are based on The Partners in Change Handbook: A Professional Development Curriculum in Mathematics, *developed under the direction of Principal Investigator Suzanne Chapin at Boston University in 1997. Preparation of the handbook was supported by the U.S. Department of Education.*

Part B: False Position and Backtracking

(45 MINUTES)

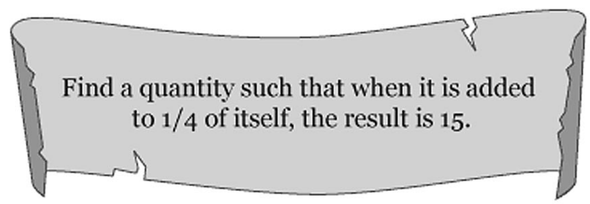
False Position

The solutions to mathematical problems often involve writing and solving equations. We generally think of solving equations by representing quantities with variables. Early Egyptians did not have symbolic notation, however, so they invented a technique called *false position* to solve mathematical problems. **[SEE NOTE 4]**

The Egyptians used false position to solve number puzzles. A modified version of false position was used by Diophantus to solve problems with squared variables and was still taught in the Renaissance for proportion problems.

False position begins by selecting a convenient answer or making an educated guess, one that makes the calculations of the problem simpler. It does not have to be the correct answer. After calculating the result from the convenient answer, a false position problem is solved by using the result to determine how to adjust the convenient answer to make it correct.

Here's an example from the Rhind papyrus, an ancient Egyptian scroll containing mathematical tables and calculations:



To use the method of false position, start by selecting a convenient answer. Let's pick 4. Why 4? It simplifies the calculation: 4, plus 1/4 of itself (which is 1), equals 5.

Next, use the result to determine how to adjust the convenient answer. We got 5 as our result, and we wanted 15. The number to multiply by 5 (the result we got) to get 15 (the result we want) is 3.

So, multiply 4 (the convenient answer we started with) by 3 to get the correct answer, which is 12.

There are a number of similar false position problems in the Rhind papyrus. Try to solve the following problem using the method of false position:

Problem B1. A quantity and its 1/7 added together become 32. What is the quantity? **[SEE TIP B1, PAGE 162]**

Problem B2. Consider the following equation: $4[3(2n - 4) / 6] = 8$. Think of as many strategies as you can for solving this equation. **[SEE TIP B2, PAGE 162]**

NOTE 4. In this section, we'll look at informal strategies for solving equations. The method of false position is introduced as a specific case of guess, check, and improve. We will also attempt to solve equations by backtracking. **Groups:** Start off by discussing the method of false position, using the historical background and the example given in the course.

The method of false position is a case of guess, check, and improve. Consider why it works, and if it works in every situation. Groups can work in pairs on Problems B1 and B2.

Part B, cont'd.

Backtracking

Here is the equation from Problem B2: 4[3(2*n* - 4) / 6] = 8. One method for solving Problem B2 is to make a flow chart of the operations on the variable. For example, the first operation on the variable n in Problem B2 is multiplication by 2. **[SEE NOTE 5]**

Problem B3. Using the equation from Problem B2:

a. Create a flow chart for the equation. Consider *n* as the input and 8 as the output.

b. Work backwards from your flow chart to find the value of *n* that produces 8 as an output. Does this method have anything in common with the method of covering up?

This method of solving equations is called *backtracking*. Backtracking involves "undoing" operations to work backwards from the output to the input.

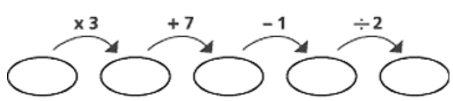
VIDEO SEGMENT (approximate times: 9:09-11:16): You can find this segment on the session video approximately 9 minutes and 9 seconds after the Annenberg/CPB logo. Zero the counter on your VCR clock when you see the Annenberg/CPB logo.

In this video segment, Professor Cossey draws a flow chart for the equation in Problem B2, then demonstrates the method of backtracking. You can choose to do Problem B3 before or after watching the video segment. You can first try to do the problem on your own, then use the video segment to reinforce what you've learned. Or you can watch the video segment before doing the problem to help you get started making flow charts or doing the method of backtracking.

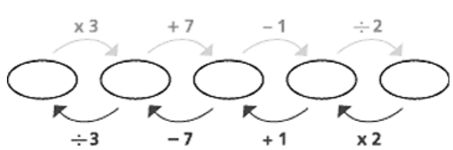
How are flow charts similar to the function machines you created in Session 3, Part C? (See page 65.)

NOTE 5. Now move on to other strategies for solving equations.

Groups: A facilitator or another group member should lead this activity. The leader begins by asking one person to think of a number. That person should write the number on a piece of paper and hold it up for the rest of the group to see, without showing it to the leader. Then, the leader asks the group to volunteer different operations with numbers, such as multiply by 2, subtract 4, add 1, etc. The leader writes these on the board as a flowchart, including about five or six steps. The picture will look something like this:



The leader then asks the group to run their number through the flowchart and only reveal their answer. The group can discuss their ideas for figuring out the input. The leader then works backwards, generating a drawing on the board that looks something like the next illustration.



Backtracking is a method that can be used before students know anything about formal equation solving. It simply requires that any operations be "undone" to work backwards from the output to the input. **Groups:** Work in pairs on Problems B3-B10. In Problem B4, recognize that in order to solve a problem using backtracking, the variable must appear on one side of the equation by itself. This is a limitation that points to the need for alternate methods.

Problem B10 is an example of a problem that is much easier to solve using backtracking. **Groups:** Share your thinking on this problem, and see if anyone tried to solve it in a different way.

Part B, cont'd.

Problem B4. Solve each of the following using backtracking:

a. $5(b / 2 - 3) = 20$

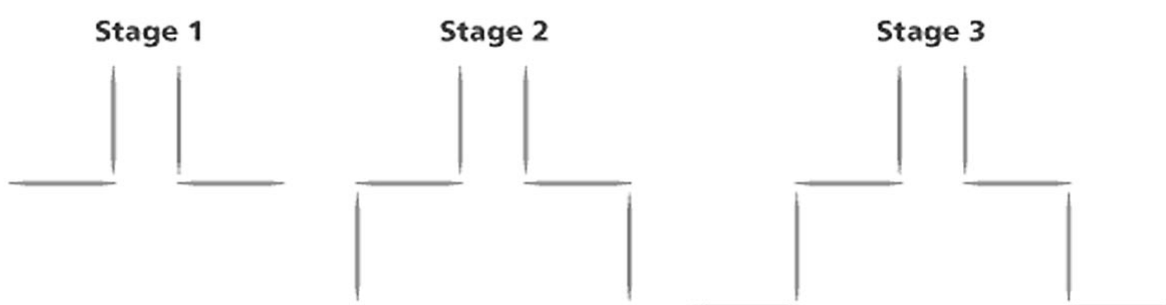
b. $7(n + 1) / 2 = 14$

Problem B5. Can you find an equation that cannot be solved by backtracking? [SEE TIP **B5,** PAGE **162**]

Problem B6. I'm thinking of a number. When I subtract 3 from my number, multiply the result by 8, then divide this result by 3, I get 16. What is my number?

Problem B7. Do problems that *can* be solved by backtracking have anything in common?

Problem B8. Look at the toothpick pattern below. One of the stages needs 112 toothpicks to form the pattern. Can you use backtracking to find out which stage it was? [SEE TIP **B8,** PAGE **162**]



Part B, cont'd.

Take It Further

Some equations lend themselves to a process called *covering up.* Covering up takes a complex equation and changes it into a series of one-step equations. For example, let's say we wanted to solve the equation:

$$\frac{21}{x + 1} - 6 = 1$$

A solution by covering up would begin by covering the most complicated expression in the equation (in this case, 21 / (x + 1) is the expression). Then the equation reads (*covered*) - 6 = 1, an equation that is solved quickly. Now we know that 21 / (x + 1) = 7. To continue, cover up the most complicated expression in the new equation, which is x + 1. The equation reads 21 / (*covered*) = 7. Now we know that x + 1 = 3, so x must be 2.

$$\frac{21}{x + 1} - 6 = 1 \;\Rightarrow\; \boxed{?} - 6 = 1 \;\Rightarrow\; \frac{21}{x + 1} = 7 \;\Rightarrow\; \frac{21}{\boxed{?}} = 7$$

Problem B9.

a. Solve Problem B2 by the method of covering up: 4[3(2n - 4) / 6] = 8

b. Solve the following equation by covering up: 3(12 / [x - 5]) + 1 = 13

Problem B10. On Monday, the produce manager stocked his store's display case with 80 heads of lettuce. By the end of the day some heads of lettuce had been sold. On Tuesday, the manager counted the number of heads of lettuce that were left and decided to add an equal number of heads of lettuce, thereby doubling the leftovers. By the end of the day, he had sold the same number of heads of lettuce as on Monday.

On Wednesday, the manager decided to triple the number of heads of lettuce that had been left in the case. He sold the same number of heads of lettuce that day, too. At the end of the day, though, there were no heads of lettuce left.

How many were sold each day? Describe the strategies you used to solve this problem. **[SEE TIP B10, PAGE 162]**

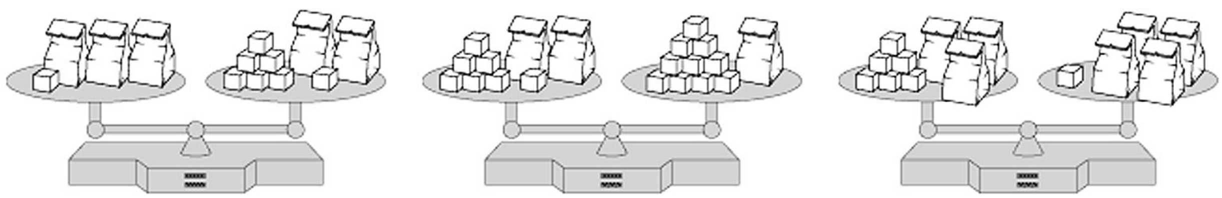
Problem B10 is taken from the Math Forum Project: Algebra Problem of the Week, posted March 29, 1999. Available online at http://mathforum.com.

Part C: Bags, Blocks, and Balance

(30 MINUTES)

Balance Scale Activity

Below are three balanced scales. Use your intuition about balance to determine, for each case, how many blocks have to be in the bag.



What would happen if you removed a block from the left? What would happen if you added a block to the left?

For the purposes of the problems, the bags in each problem hold the same number of blocks, and the bags themselves weigh practically nothing.

Try It Online!

This problem can be explored online as an Interactive Activity. Go to the ***Patterns, Functions, and Algebra* Web site** at www.learner.org/learningmath and find Session 6, Part C, Balance Scale Activity.

VIDEO SEGMENT (approximate times: 13:31-14:36): You can find this segment on the session video approximately 13 minutes and 31 seconds after the Annenberg/CPB logo. Zero the counter on your VCR clock when you see the Annenberg/CPB logo.

In this video segment, Sue-Anne tells Professor Cossey how she solved a balance puzzle while Professor Cossey follows along on a chart. Watch the video segment after you have completed the Balance Scale activity and compare your strategy with Sue-Anne's. If you get stuck during the activity, you can watch the video segment to help you.

Was Sue-Anne's thinking similar to or different from your own strategy for these problems? Will Sue-Anne's strategy work for any balance puzzle? Will yours? Explain.

Problems in Part C are taken from IMPACT Mathematics Course 2, *developed by Education Development Center, Inc. (New York: Glencoe/McGraw-Hill, 2000), p. 389.*

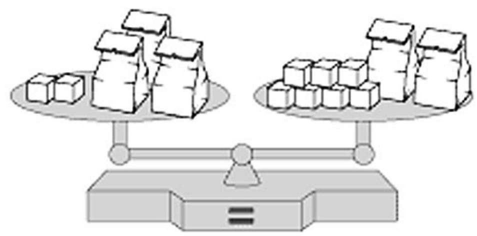
Part C, cont'd.

Balance Puzzles and Equations

You probably noticed in the Balance Scale activity that in order to keep balance, you must do the same thing to both sides of the scale. In an algebraic equation, the balance is represented by the equal sign, and only doing the same thing to both sides of the equation will preserve the balance. **[SEE NOTE 6]**

Problem C1. Make up your own bag and block balance puzzle. If you are working with someone else, exchange puzzles with your partner and solve your partner's puzzle.

Problem C2. Look at the balance puzzle at right.



a. Assign a variable to the number of blocks in each bag. Write an equation that fits the puzzle.

b. Using the drawing, find the number of blocks in each bag.

c. Solve the equation by doing the same thing to both sides. How is the solution to the equation related to what you did in part (b) of this problem? **[SEE TIP C2, PAGE 162]**

Problem C3. Create a balance puzzle where the solution is not a whole number of blocks. How could you modify your model to fit this solution? **[SEE NOTE 7]**

NOTE 6. Now let's look at a new model for solving equations, one that works in situations where backtracking doesn't.

The model of bags and blocks focuses on the idea of balance. Each bag holds an unknown number of blocks (eventually represented by a variable). Each bag in the same problem must hold the same number of blocks, just as a variable in an equation must stand for the same number every time it is used.

The challenge is to solve a series of problems by figuring out how many blocks are in each bag. Try to solve the problems by removing the same thing from each side of the scale, which is comparable to the more formal process of doing the same thing to both sides of an equation. **Groups:** Work in pairs on Problems C1-C7.

NOTE 7. Problem C3 brings up the interesting question of what happens when the solution is not a whole number. **Groups:** Some might say that it's possible to change the model so that the blocks are made out of some material, like clay, that can be cut up into fractions of blocks. Others might say that the model cannot be used. There is no need for consensus on this issue—the point is that when models fail, they can either be modified or abandoned for new methods.

Part C, cont'd.

Problem C4. Draw a balance puzzle that represents $2h + 3 = h + 8$. Now solve the balance puzzle. In the puzzle, what is represented by the h in the equation?

Problem C5. Draw a balance puzzle that represents $3b + 7 = 3b + 2$. Now solve the equation. Explain what happens. Which equation below (from Problem A1) is most like this one? **[SEE NOTE 8]**

a. $5 + 3 = 8$	b. $2 + 14 = 12$	c. $5 + 3 = y$
d. $x + 3 = y$	e. $3x = 2x + x$	f. $3x = 3x + 1$

Problem C6. Draw a balance puzzle that represents $4b + 3 = 4b + 3$. How is this different from what happened in Problem C5? Which equation above is most like this one?

Problem C7. Can you draw a balance puzzle to represent the equation $4x - 2 = 5x - 3$? Why or why not? **[SEE NOTE 9]**

Problem C7 brings out some of the limitations of the balance model. The method of doing the same thing to both sides may still be used to solve problems that are difficult to represent with balance puzzles.

NOTE 8. Problem C5 revisits an idea from the previous session: equations that have no solution. We have seen this idea modeled as two parallel lines: no intersection means no solution. This is another way to think about the same situation. Taking an equal number of bags off each side leaves unequal numbers of individual blocks, which would not balance each other. Thus, the original equation was never really in balance to begin with.

NOTE 9. The representation of negative blocks in Problem C7 is even more difficult and is, in fact, a good example of when a model becomes more trouble than it's worth. Many models outgrow their usefulness in this way.

Part C, cont'd.

Problem C8. Solve the equation in Problem C7 by doing the same thing to both sides.

Problem C9. One method of teaching how to solve equations is that "if you don't like which side a number is on, move it to the other side and switch the sign." How is this related to the method of doing the same thing to both sides?

VIDEO SEGMENT (approximate times: 17:42-18:22): You can find this segment on the session video approximately 17 minutes and 42 seconds after the Annenberg/CPB logo. Zero the counter on your VCR clock when you see the Annenberg/CPB logo.

In this video segment, Sue-Anne says that the balance puzzle helped her see why it was so important to do the same thing to both sides, and emphasized the importance of using this analogy with her students.

Think about the problems you've worked on so far in this session. Will the method of doing the same thing to both sides solve every problem in this session, or just some of them?

There are many strategies that people use to solve equations: guessing and checking, backtracking or inverting operations, and doing the same thing to both sides. For any particular problem, one method may be easier than another. The word "easier," however, has two different meanings. It might mean "more conceptually understandable," or it might mean "more efficient to compute." The guess-and-check method is rarely efficient, but students understand it. Backtracking or inverting operations doesn't always work. Doing the same thing to both sides always works, but sometimes the computation is messy.

Homework

In the puzzles that follow, each shape stands for a number value of "weight." To solve these puzzles, make the weights in each part of the mobile balance from left to right, just as a sculptor might balance all the parts of a mobile. Here are the rules:

a. The right and left sides of each horizontal beam must balance.

b. Each shape has a unique and consistent weight within a puzzle, and no shapes weigh 0.

c. Take the clues at face value. For example, if a clue says that the square's weight is a multiple of the triangle's weight, you can assume that the triangle does not weigh 1.

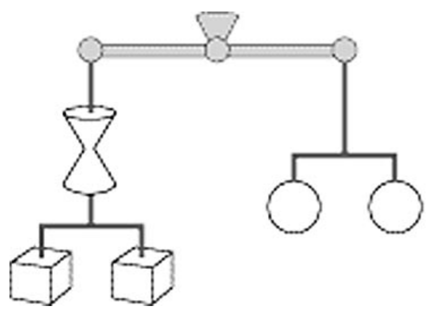
d. All weights are either one- or two-digit, positive whole numbers.

e. A piece hanging directly below the fulcrum does not affect the balance between the left and right arms. Although this piece has its own definite weight, it remains "neutral" for the purpose of balancing the other two arms.

f. The sizes of the pieces have no relation to weight.

g. These mobiles are exercises in balancing number values. They do not take into account the distance from the fulcrum. **[SEE TIP H1-H4, PAGE 162]**

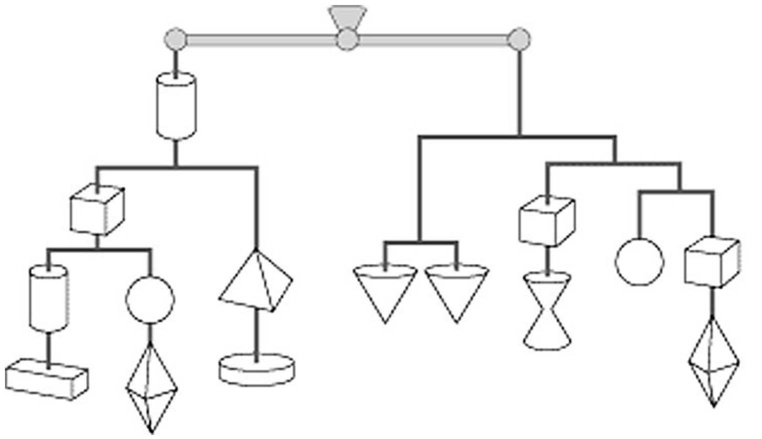
Problem H1. Discover the value of each of the shapes. The total weight is 36. All shapes weigh less than 10.

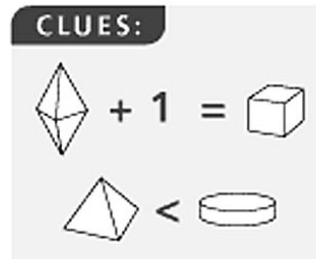


The homework problems are excerpted from In the Balance, *by Lou Kroner (New York: Creative Publications, Wright Group/McGraw-Hill, 2000). The materials may not be reproduced without written permission of Creative Publications.*

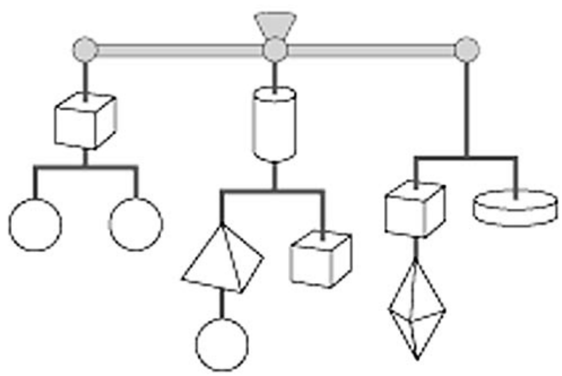
Homework, cont'd.

Problem H2. Discover the value of each of the shapes. The total weight is 80. Only one shape weighs more than 9.



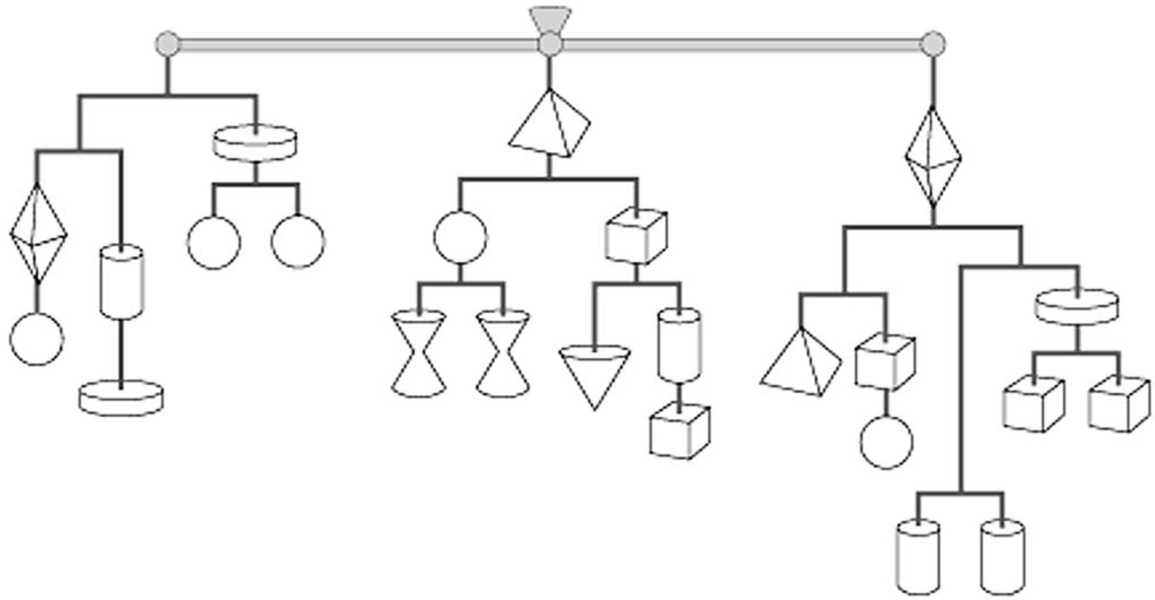


Problem H3. Discover the value of each of the shapes. The total weight is 54. The three arms are equal in weight.



Homework, cont'd.

Problem H4. Discover the value of each of the shapes. The total weight is 180. Each of the three arms is equal in weight.



Problem H5. Marcus had some cookies. He wanted to give them all away. He gave $^1/_2$ of them to his friend David. They divided the remaining cookies evenly among David's 3 brothers, so each got 4. How many cookies did Marcus have originally? **[SEE TIP H5, PAGE 162]**

Suggested Reading

These readings are available as downloadable PDF files on the *Patterns, Functions, and Algebra* Web site. Go to:

www.learner.org/learningmath

Stacey, Kaye. "Ideas About Symbolism That Students Bring to Algebra." *The Mathematics Teacher* 90 (February 1997).

Tips

Part A: Equality and Balance

TIP A2: The definition of a solution set is helpful here. Consider how many different solutions there are in each solution set.

TIP A5: Think about the strategy Frederick used in the video segment. Additionally, think about what you can do to each side of a scale while keeping it balanced. Any such step can be done to any of the scales.

Part B: False Position and Backtracking

TIP B1: Start by assuming a "convenient" value for the solution, one that makes the "$^1/_7$" portion of the problem easier. Then, decide how to adjust the answer you got to make 32, the answer you want. "Its $^1/_7$" means $^1/_7$ of the original quantity.

TIP B2: False position will work here. You might also try writing the steps of the equation as an algorithm of function machines, and use a diagram to find the value of *n*. (See Session 3, Part C, page 65.)

TIP B5: Think about whether a function machine's operation could always be "undone." You might also look for problems in Part C of this session that would be difficult to solve by backtracking.

TIP B8: A similar problem appears in Session 2, Part B (see page 40). If you find a formula for the number of toothpicks at a given stage, you can backtrack or cover up using that formula.

TIP B10: Which of the methods in this part would be useful here? What would be a useful variable? Don't forget: The same number of heads of lettuce are sold each day.

Part C: Bags, Blocks, and Balance

TIP C2: If you are having trouble here, watch the video segment indicated on page 155.

Homework

TIP H1-H4: Think about the total weight of each branch of the mobile, which may give you the values of certain weights or help you find equations relating weights. In tougher puzzles, make a list of possible values for a weight, then try to reduce that list to one correct value. Problem H4 is a system of equations with eight variables, and is very difficult.

TIP H5: Which of the methods of solving equations that you have learned in this session can be useful here? There is more than one answer.

Solutions

Part A: Equality and Balance

Problem A1.

a. This equation is always true.

b. This equation is always false.

c. This equation is true only if $y = 8$. It is false if y is any other number.

d. This equation is true for an infinite number of pairs of values for x and y. For example, $x = 1$ and $y = 4$ make the equation true, while $x = 6$ and $y = 3$ make it false.

e. This equation is always true.

f. This equation is always false.

Problem A2.

a. There are no variables in this equation, so there is no variable to solve for. This equation is always true.

b. There are no variables in this equation, so there is no variable to solve for. This equation is always false.

c. The solution set is {8}, because $y = 8$ is the only number that would make the equation true.

d. There are an infinite number of pairs of solutions: $x = 1$, $y = 4$ is one. The solution set is {(1, 4), (2, 5), (3, 6), (4, 7), ...}, although there are many other solutions which are not integers; ($1\ ^1/_2$, $4\ ^1/_2$) is one.

e. There are an infinite number of solutions—x can be any number, and the equation will always be true. The solution set is {..., -3, -2, -1, 0, 1, 2, 3, ...}. As in the previous question, there are many more solutions that are not integers.

f. There are no solutions to this equation, because the right side is always 1 larger. The equation is always false.

Problem A3.

a. One answer is to replace "?" with 31. Another is to replace it with 21 + 10 or any quantity that must equal 31.

b. Replace "?" with 32, or any quantity not equal to 31.

c. Replace "?" with a variable letter, like x.

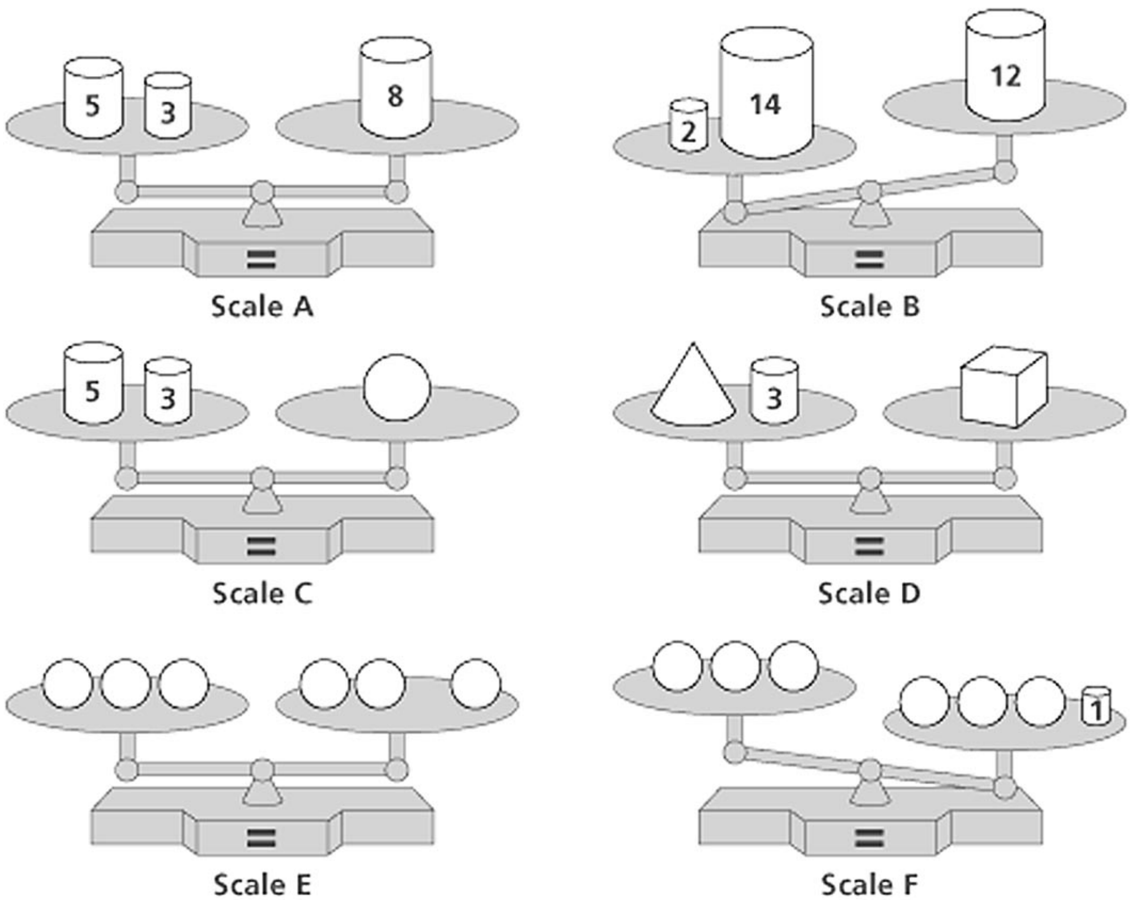
Problem A4. The solution is that 1 rectangle will balance with 1 square. While it would also be correct to say that 1 rectangle will balance with $^1/_3$ of a circle, this answer does not reflect use of the second scale.

Problem A5. One possible answer is that, for scale D, the circle is equivalent to 5 cubes. The cone is equivalent to 4 cubes, and the cylinder is equivalent to 6 cubes.

Problem A6. One possible answer is that, for scale D, the cylinder and circle are equivalent to 1 cube. The cylinder is equivalent in weight to 3 circles, and the cube is equivalent to 4 circles.

Solutions, cont'd.

Problem A7. Each scale is balanced only if the quantities on each side of the scale are equivalent. In the same way, an equation is true only if the quantities on each side of the equal sign are equivalent. When variables are used, the scale may be balanced always [as it is in equation (e)], sometimes [as in equations (c) and (d)], or never [as in equation (f)].



Part B: False Position and Backtracking

Problem B1. First, choose a convenient answer. Because we are to multiply the answer by 1/7, use 7: $1\frac{1}{7}$ times 7 is 8. The number to multiply by 8 to get 32 is 4. Then multiply 7 by 4 to get the correct answer, 28. By way of checking, $\frac{1}{7}$ of 28 is 4, and 28 + 4 = 32, which is the answer we wanted.

Problem B2. We could use false position. We could guess-and-check. We could begin multiplying and dividing from the inside out to produce a more easily solvable equation. We could build an operations flowchart for the equation, then follow it backwards to see what input value would lead to 8 as the output. We could begin dividing and multiplying, from the outside in, in the hope that cancellation will lead to a more easily solvable equation. We could build a table of values with a spreadsheet in the hope that a pattern will emerge. We could use a graphing calculator to graph the left and right sides (right side: $y = 8$) of the equation, then look for the intersections.

Solutions, cont'd.

Problem B3.

a. The flowchart steps are multiply by 2, subtract 4, multiply by 3, divide by 6, and multiply by 4.

b. Working backwards from 8, we divide by 4 to get 2, multiply by 6 to get 12, divide by 3 to get 4, add 4 to get 8, and divide by 2 to get 4 as the correct value of n. Testing $n = 4$ in the equation shows that this is correct. Notice that the steps we perform are the same steps we perform when covering up.

Problem B4.

a. In the original order, the steps are divide by 2, subtract 3, and multiply by 5. So, take 20, divide by 5 to get 4, add 3 to get 7, and multiply by 2 to get 14 as the answer. Testing $b = 14$ confirms it as correct.

b. In the original order, the steps are add 1, multiply by 7, and divide by 2. So, take 14, multiply by 2 to get 28, divide by 7 to get 4, and subtract 1 to get 3 as the answer. Test it out!

Problem B5. Sure, such equations abound, so long as you can find an operation that can't be "undone." An equation like $b^2 - 5b = 6$ does the trick nicely—most equations that involve squares cannot be undone in a straightforward manner. Also, any equation that has a variable on both sides of the equal sign, like those in Part C, cannot be easily solved by backtracking.

Problem B6. Undo the steps. Take 16 and multiply by 3 (undoing the last step first) to get 48. Then divide 48 by 8 to get 6. Finally, add 3 to get 9, the original number. Test 9 to check that it is correct.

Problem B7. Each of these has two things in common: the variable being solved for only occurs once, and steps in the equation that can be reversed. Since backtracking is so similar to "undoing" the effect of several function machines, any step in the equation must be reversible for backtracking to be successful.

Problem B8. The first stage has 4 toothpicks, and each stage beyond the first needs 2 more toothpicks. One way of backtracking is to use the fact that we need to add 108 toothpicks from the initial formation, and dividing 108 by 2 (= 54) will tell us how many stages to add. Because there are 4 toothpicks in the first stage, 54 more stages will bring us to the 55th stage, which has 112 toothpicks.

Alternately, we can find a formula for the number of toothpicks in terms of the stage number. One such formula is $T = 2n + 2$ (another is $T = 2(n - 1) + 4$). Then use the techniques of Problem B4 to find n, which is 55.

Problem B9.

a. Cover up everything inside the brackets to get 4(*covered*) = 8. So $3(2n - 4) / 6 = 2$. Cover up $3(2n - 4)$ to get (*covered*)/ 6 = 2. Now $3(2n - 4) = 12$. Cover up $(2n - 4)$ to get 3(*covered*) = 12. So $2n - 4 = 4$. Then cover up $2n$ to get (*covered*) - 4 = 4. So $2n = 8$, and $n = 4$ is the solution.

b. Cover up $3(12 / [x - 5])$ so the equation reads (*covered*) + 1 = 13. Then $3(12 / [x - 5]) = 12$. Then cover up $(12 / [x - 5])$ to get 3(*covered*) = 12. So $12 / [x - 5]= 4$. Finally, cover up $(x - 5)$ to get the equation 12 / (*covered*) = 4. From here we get the equation $x - 5 = 3$, for which $x = 8$ is the solution.

Problem B10. One of the best ways to negotiate this problem is to use a variable for the "unknown," the number of heads of lettuce sold each day. Call this number n. Now we can follow the problem either forward or backward.

Following it forward...

Start of first day: 80
End of first day: $80 - n$
Start of second day: $2(80 - n) = 160 - 2n$
End of second day: $160 - 2n - n = 160 - 3n$
Start of third day: $3(160 - 3n) = 480 - 9n$
End of third day: $480 - 9n - n = 480 - 10n$. Since this equals 0, $480 - 10n = 0$, and this means $480 = 10n$ and $48 = n$.

Following it backward...

End of third day: 0
Start of third day: n
End of second day: $1/3\ n$ [the stock was tripled!]
Start of second day: $1/3\ n + n = 1\ ^1/_3\ n$
End of first day: $2/3\ n$ [half of $1\ ^1/_3$]
Start of first day: $2/3\ n + n = 1\ ^2/_3\ n = 80$ heads. So $n = 80$ divided by $1\ ^2/_3 = 48$.

Notice that backtracking seems to be a little easier here. A solution obtained by covering up is possible, but it would be more difficult because it would require you to write an equation for the entire problem in which there is only one variable.

You could also use false position and decide whether the number of heads sold in a day was too small (if some were left at the end), too large (if not enough were left to be sold on the third day), or just right (48). In this problem, though, false position amounts to guess-and-check, because there is no easy way to adjust a wrong answer to make a right one.

Part C: Bags, Blocks, and Balance

Problem C2.

a. Let n = the number of blocks in each bag. Then the equation is:

   $3n + 2 = 2n + 7$

   because the two sides of the scale must be equivalent in weight.

b. Remove 2 bags from each side. Then remove 2 blocks from each side. There will now be 1 bag on the left and 5 blocks on the right, so it must be that there are 5 blocks in each bag.

c. Subtract $2n$ from each side (remove 2 bags from each side) to yield $n + 2 = 7$. Then subtract 2 from each side (remove 2 blocks from each side) to yield $n = 5$. This means that each bag contains 5 blocks.

Problem C3. One such problem has 3 bags and 2 blocks on the left, and 1 bag and 7 blocks on the right. Solving yields that each bag should have $2\ ^1/_2$ blocks. One way to modify the model is to have the solution represent the relative weight of the bag rather than the number of blocks inside it.

Problem C4. The solution is $h = 5$. In the balance puzzle, this means that there would be 5 blocks in each bag, since h was the variable assigned to the unknown quantity.

Problem C5. There is no solution! One explanation is that no matter what value b represents, the left side of the equation will be larger than the right. Looking at the balance puzzle, there are always 5 more blocks on the left side than on the right. This is most similar to equation (f) of Problem A1, which had no solution.

Problem C6. Every number is a solution! No matter what value b represents, the left side and right side are equal. In the balance puzzle, no matter how many blocks are in each bag, the balance is maintained. Note that this means there is *not enough information* to find the value of b. This is most similar to equation (e) of Problem A1, which had an infinite number of solutions.

Problem C7. It can be done, but it requires some changes. One way is to represent "subtracting 2" by adding 2 blocks to the opposite side. A better way is to use items that would reduce the weight of a side, such as "negative blocks" or helium balloons. Each balloon cancels the weight of a block, so adding a block to a side would be identical to removing a balloon, and vice versa. Note that in this problem, x represents the number of blocks in each bag.

Solutions, cont'd.

Problem C8. Add 3 to each side to yield $4x + 1 = 5x$. Then, subtract $4x$ from each side to yield $1 = x$, the solution.

Problem C9. Consider an equation like $3x + 5 = 23$. Solving this by moving 5 to the right and switching the sign gives $3x = 23 + (-5)$, or $3x = 18$. Solving by doing the same thing to both sides would require us to subtract 5 from both sides:

$$\begin{array}{r} 3x + 5 = 23 \\ -5 \quad -5 \\ \hline 3x = 18 \end{array}$$

Note that this is identical in result to the solution by moving, but it also explains the origin of the -5.

Homework

Problem H1. The circles must each be 9, because the mobile splits twice. The clue tells us the cubes cannot weigh more than 5 each. Because the hourglass and 2 cubes weigh 18 and the hourglass weighs less than 10, each cube is 5 and the hourglass is 8.

Problem H2. The upside-down cone is 10 (2 splits), and the circle is 5 (3 splits). Because the cube and the diamond make 5 together, the clue tells us the diamond is 2, and the cube is 3. The others: hourglass = 7, cylinder = 6, horizontal bar = 1, pyramid = 8, flat disk = 9.

Problem H3. The flat disk must be 9 (2 splits). The cube must be an even number, and larger than the circle—this makes it 8. The others: circle = 5, pyramid = 3, cylinder = 2, diamond = 1.

Problem H4. One possible course is to find the value of the diamond first. After the diamond in the 3rd set, there are 3 consecutive splits (to the cylinders), so the total weight after the diamond must be a multiple of 8. This limits the possible values of the diamond shape to 4 or 12, because the diamond on the left, when paired with a circle, weighs a total of 15. Investigating the circles in the left branch shows that the circle must be 11, and the diamond must be 4. The others: cylinder = 7, flat disk = 8, pyramid = 14, hourglass = 6, cube = 3, upside-down cone = 10.

Problem H5. Backtrack or work from false position.

Backtracking: Each brother got 4 cookies, so the brothers shared 12. This is half of the original number of cookies, so the original number is 24 (multiply 12 by 2).

False position: Pick a convenient number of cookies for Marcus, then see what would have happened. Say that Marcus had 60 cookies (a good number, since 60 is very divisible). Half go to David, so there are 30 to split among 3 brothers. If Marcus had 60, each brother gets 10 cookies.

So, to turn 10 into 4 we need to multiply by 0.4, which shows that Marcus had (60)(0.4) = 24 cookies to start.

As with Problem B10, finding the solution by covering up is possible, but difficult. A solution found by doing the same thing to both sides is only possible after creating an equation for the entire situation.

Notes