Session 10

Classroom Case Studies, Grades 3-5

This is the final session of the ***Patterns, Functions, and Algebra*** course! In this session, we will examine how the types of mathematical tasks involving algebraic thinking from the previous nine sessions might look when applied to students in your own classrooms. This session is customized for three grade levels. Select the grade level most relevant to your teaching.

The session for grades 3-5 begins below. Go to page 251 for grades K-2 and page 293 for grades 6-8.

Key Terms for This Session

Previously Introduced

- mathematical thinking tools [Session 1]
- representation [Session 1]
- input [Session 2]
- recursive description [Session 2]
- function [Session 3]
- algebraic ideas [Session 1]
- variable [Session 2]
- output [Session 2]
- closed-form description [Session 2]
- backtracking [Session 6]

Introduction and Review

In the previous sessions, we explored the components of algebraic thinking: fundamental algebraic ideas (content) and mathematical thinking tools (process). You were asked to put yourself in the position of a mathematics learner, both to analyze your individual approach to solving problems and to get some insights into your own conception of algebraic thinking. It may have been difficult for you to separate yourself as a mathematics learner from yourself as a mathematics teacher. Not surprisingly, this is often the case! In this session, however, we will shift the focus to your classroom and to the approaches your students might take to mathematical tasks involving algebraic thinking. **[SEE NOTE 1]**

Learning Objectives

In this session, we will focus on your experiences as a classroom teacher, as you:

- Explore how algebraic thinking is developed at your grade level
- Examine problems for their algebraic content
- Analyze mathematical tasks and their connection to mathematical themes in the course
- Critique lessons at your grade level for algebraic thinking

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NOTE 1. This session focuses on developing algebraic thinking in the grades 3-5 classroom. We'll consider the mathematics content of the previous sessions and the relation of that content to the mathematics you teach in your own classrooms. We'll explore how algebraic thinking develops at your grade level by analyzing mathematical tasks appropriate for the grades 3-5 classroom. We'll also look at lessons from existing curriculum materials and critique them in relation to how students are asked to demonstrate their thinking and how the mathematics reveals algebraic thinking.

NOTE 1 cont'd. next page

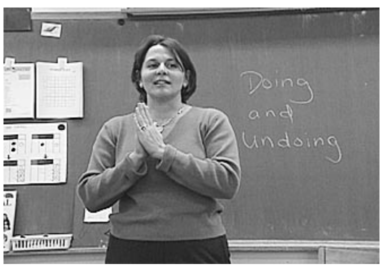
Part A: Classroom Video (30 MINUTES)

To begin the exploration of what algebraic thinking looks like in a classroom at your grade level, watch a video segment of a teacher who has taken the *Patterns, Functions, and Algebra* course and has adapted the mathematics to her own teaching situation. When viewing the video, keep the following three questions in mind: **[SEE NOTE 2]**

a. What fundamental algebraic ideas (content) is the teacher trying to teach? Think back to the big ideas of the previous sessions: patterns, functions, linearity, proportional reasoning, nonlinear functions, and algebraic structure.

b. What mathematical thinking tools (process) does the teacher expect students to demonstrate? Think back to the processes you identified in the first seminar: problem solving skills, representation skills, and reasoning skills.

c. How do students demonstrate their knowledge of the intended content? What does the teacher do to elicit student thinking?



VIDEO SEGMENT (approximate times: 5:43-11:08): You can find this segment on the session video approximately 5 minutes and 43 seconds after the Annenberg/CPB logo. Zero the counter on your VCR clock when you see the Annenberg/CPB logo.

In this video segment, Liza Jones introduces her students to the process of undoing. She begins by asking her students to figure out the "recipe" or formula she is using to turn an In into an Out. She then asks them to undo the "recipe" by finding the In when given the Out.

Problem A1. Reflect on questions (a), (b), and (c) above. **[SEE NOTE 3]**

NOTE 1, CONT'D.

Review

Begin the session by reviewing the mathematics content in the previous nine sessions: patterns, functions, linearity, proportional reasoning, nonlinear functions, and algebraic structure. You may want to think about one big idea in each of these topics.

Homework Review

Groups: Discuss any questions about the homework.

NOTE 2. Before examining specific problems at your grade level with an eye toward algebraic thinking, we'll watch another teacher—one who has also taken the course—teaching in her classroom. The purpose is not to be critical of the teacher's methods or teaching style. Instead, look closely at how the teacher brings out algebraic ideas in teaching the topic at hand, as well as how the teacher extends the lesson and asks questions that elicit algebraic thinking.

Review the meaning of algebraic ideas (that is, the content of algebra) and mathematical thinking tools (the processes used in analyzing problems). Keep in mind questions (a), (b), and (c) as you watch the video.

NOTE 3. Groups: Work in small groups on Problems A1-A4. Share answers to Problem A4 especially, because recursive thinking is not necessarily something that third-grade teachers often consider in their teaching. The vocabulary (recursive) is not the important part here. The method of thinking recursively, however, is important for teachers to consider as students make sense of patterns.

Part A, cont'd.

Problem A2. How does Ms. Jones incorporate the concept of doing and undoing into the model of function machines?

Problem A3. How do the students think about undoing the recipe they constructed?

Problem A4. How could Ms. Jones have used recursive thinking in this lesson?

Part B: An Example for Developing Algebraic Thinking (20 MINUTES)

The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (2000) identifies algebra as a strand for grades Pre-K-12. The *Standards* identify the following concepts that all students should cover and comprehend: **[SEE NOTE 4]**

- Understand patterns, relationships, and functions
- Represent and analyze mathematical situations and structures using algebraic symbols
- Use mathematical models to represent and understand quantitative relationships
- Analyze change in various contexts

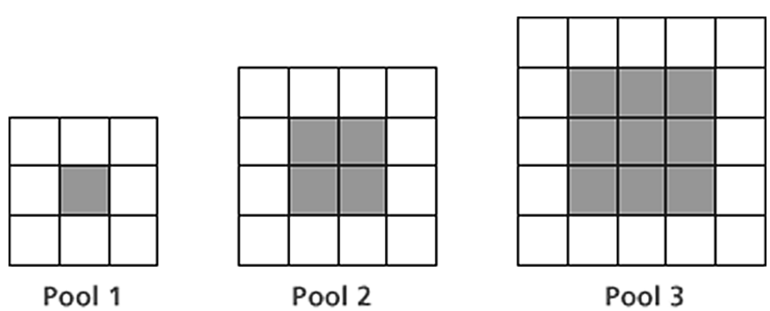
For the classroom in Grade 3-5, understanding patterns includes the following expectations:

- Describe, extend, and make generalizations about geometric and numeric patterns
- Represent and analyze patterns and functions, using words, tables, and graphs
- Represent the idea of a variable as an unknown quantity using a letter or a symbol

In this part, we'll look at problems that foster algebraic thinking as it relates to these standards, and we'll explore ways of asking questions that elicit algebraic thinking. The situations we will be exploring are representative of the kinds of problems you would find in some existing texts; in fact, you may recognize some of them! The goal is for you to examine these problems with the critical eye of someone who has taken this course and is beginning to view algebraic thinking with a different perspective.

Consider the situation below, appropriate for exploration in a grade 3-5 classroom:

Tat Ming is designing square swimming pools. Each pool has a square center that is the area of the water. Tat Ming uses blue tiles to represent the water. Around each pool there is a border of white tiles. Here are pictures of the three smallest square pools that he can design, with blue tiles for the interior and white tiles for the border. **[SEE NOTE 5]**



NOTE 4. Look at NCTM's recommendations for content in the algebra strand in the *Standards*, then look at the problem for designing square swimming pools. After reading the problem, work on Problems B1-B5.

NOTE 5. Read the commentary on the swimming pool problem in "Experiences With Patterning," by Joan Ferrini-Mundy, Glenda Lappan, and Elizabeth Phillips, in *Teaching Children Mathematics* (February 1997), pp. 282-288. The article can be found on the course Web site. Go to www.learner.org/learningmath and find *Patterns, Functions, and Algebra* Session 10, Grades 3-5, Part B, Note 5.

NOTE 5 cont'd. next page

Part B, cont'd.

Problem B1. What questions would you, as a mathematics learner, want to ask about this situation?

Problem B2. How do your questions reflect the algebra content in the situation?

Now focus on the questions you want the students in your classroom to consider. You may want to begin by having them draw a table that shows the pool "number" and the relative number of blue and white tiles. Students may also want to begin by looking at the fraction of blue or white tiles out of the whole number of tiles.

Problem B3. What patterns, conjectures, and questions will your students find as they work with this situation?

Problem B4. What questions could you as the teacher pose to elicit and extend student thinking at your grade level?

Problem B5. Recall the framework you explored in Session 2 in looking at patterns: finding, describing, explaining, and using patterns to predict. Which of these skills will your students use in approaching this problem?

NOTE 5, CONT'D.

Now look at the problems on patterns from *Everyday Mathematics, Student Journal*. Think of questions that extend students' thinking beyond simply finding the next number:

- Could you predict the 20th number in the list?
- Could 11 ever be a number in the list?
- If you knew the 99th number, could you determine the 101st number?

Part B, cont'd.

Problem B6. Read the article "Experiences With Patterning" from *Teaching Children Mathematics,* which can be found on the course Web site. Go to www.learner.org/learningmath and find *Patterns, Functions, and Algebra* Session 10, Grades 3-5, Part B, Problem B6. What ideas mentioned seem appropriate for your classroom?

Here is a set of problems on patterns:

37, 40, 43, ____, ____, ____

27, 25, ____, 21, ____, ____

____, 11, 15, ____, 23, ____

____, ____, 36, ____, 27

Rule						
+5¢	10¢	25¢				
Double	2		8		64	
____		7	15	19		
____			20	10	5	
Make up one of your own.						
____						

Problem B7. What questions could you ask to develop students' skills in describing these patterns?

Problem B8. What questions could you ask to develop students' skills in predicting?

The swimming pool problem is adapted from Algebra in the K-12 Curriculum: Dilemmas and Possibilities, *Final Report to the Board of Directors, by the NCTM Algebra Working Group (East Lansing. Mich.: Michigan State University 1995).*

Problem and analysis of algebraic thinking are discussed in "Experiences With Patterning," by Joan Ferrini-Mundy, Glenda Lappan, and Elizabeth Phillips, in Teaching Children Mathematics *(February 1997), pp. 282-288.*

The pattern problems are taken from Everyday Mathematics, Student Journal, Volume 1, Grade 3, *developed by the University of Chicago Math Project (New York: SRA/McGraw-Hill, 2001).*

Part C. Patterns That Illustrate Algebraic Thinking (25 MINUTES)

Many students find that a context or situation helps them think about algebraic ideas. In the next problem, we'll look at different representations of a situation that lead to a solution. **[SEE NOTE 6]**

In this section we'll analyze problems at the 3-5 grade level for their algebraic content. For each problem, find a mathematical solution, then answer questions (a) through (d) listed below.

a. How would you solve the problem?

b. What is the algebraic content in the problem?

c. How do you think your students might solve the problem? What different representations might they use?

d. What question or questions might you ask to get at "doing and undoing"?

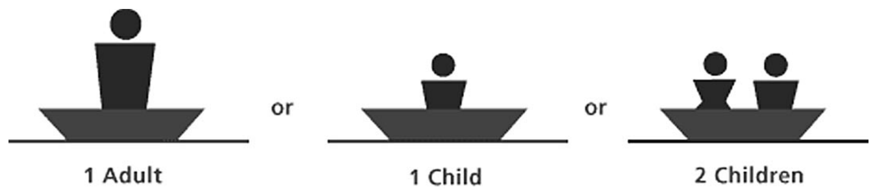
NOTE 6. Groups: Work on the crossing-the-river problem in small groups. Use concrete representations of the passengers, since "performing" the trips gives some insight into the solution of the problem.

Read "Patterns As Tools for Algebraic Reasoning," by Kristen Herbert and Rebecca Brown, in *Teaching Children Mathematics* (February 1997), focusing on different student approaches to solving the problem. This article can be found on the course Web site. Go to www.learner.org/learningmath and find *Patterns, Functions, and Algebra* Session 10, Grades 3-5, Part C, Note 6.

Part C, cont'd.

Here's the problem:

A group of 8 adults and 2 children needs to cross a river. They have a small boat that can hold either 1 adult, 1 child, or 2 children.



Problem C1. How many one-way trips does it take for the entire group of 8 adults and 2 children to cross the river? Tell how you found your answer.

Problem C2. How many trips in all for 6 adults and 2 children?

Problem C3. How many trips for 15 adults and 2 children?

Problem C4. How many trips for 23 adults and 2 children?

Problem C5. How many trips for 100 adults and 2 children?

Problem C6. Tell how you would find the number of one-way trips needed for any number of adults and 2 children to cross the river.

The river problem is taken from MathScape, *developed by Education Development Center, Inc. (New York: Glencoe/McGraw-Hill, 2000).*

Part D. More Problems That Illustrate Algebraic Thinking (25 MINUTES)

At the 3-5 grade level, there are many problems that prepare students for their later work in algebra. Experiences with these problems can also build bridges between arithmetic and algebra, while increasing the students' chances for success in future mathematics. **[SEE NOTE 7]**

In this section we'll analyze problems at the 3-5 grade level for their algebraic content. For each problem, find a mathematical solution, then answer questions (a) through (f) listed below.

a. What algebraic content is in the problems?

b. What content does it prepare students for later?

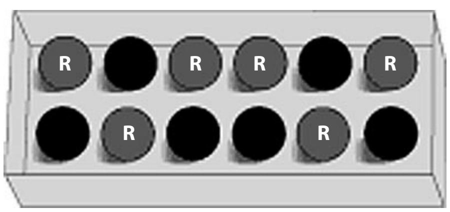
c. How does this content relate to the mathematical ideas in this course?

d. How would your students approach this problem?

e. What are other questions that might extend students' thinking about the problem?

f. Does your current program in mathematics at your school include problems of this type?

Problem D1.



There are red (R) and black marbles in a box in the drawer. Take out some marbles so that there are 3 red marbles for each black marble left in the box.

a. What is the fewest marbles you must take from the box?______

b. Draw a picture of the marbles left. Mark B or R to show the color of each marble.

c. Tell how you decided how many to take out.

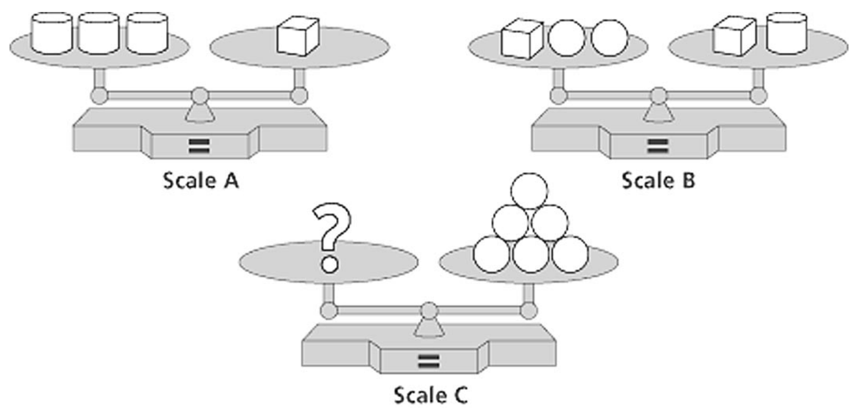
NOTE 7. It is often difficult for teachers to see how the mathematics content at their grade level builds algebraic thinking. In this part, we'll look at many examples of elementary mathematics that build bridges to algebraic thinking. **Groups:** Work on Problems D1-D5, recording answers to questions (a) through (f) as you do each problem. It is important to identify the mathematical content in each of the problems and how that content illustrates algebraic thinking.

Part D, cont'd.

Problem D2.

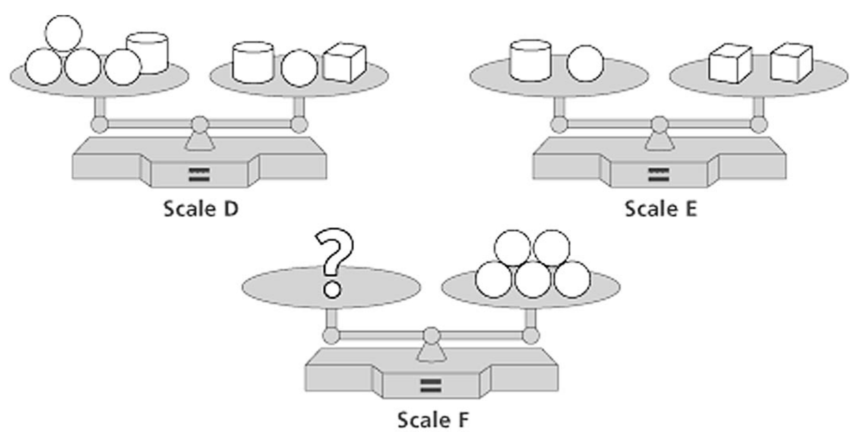
a. Which block, cube, cylinder, or sphere will balance Scale C?

b. List or draw the steps you followed to identify the block.



c. Which block, cube, cylinder, or sphere will balance Scale F?

d. List or draw the steps you followed to identify the block.



Problem D3.

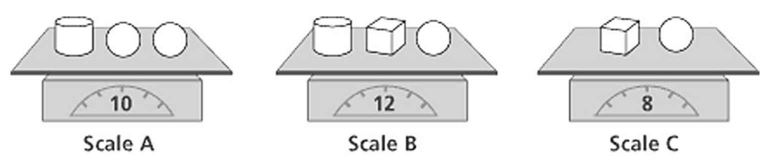
a. Find the weight of each block.

cylinder = ________ pounds

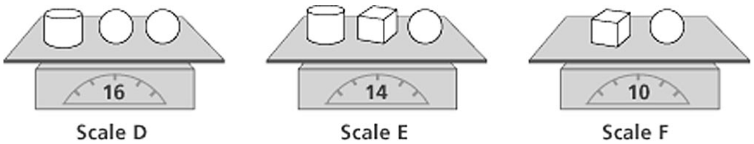
sphere = ________ pounds

cube = ________ pounds

b. List or draw the steps you followed to find the weights of the blocks.



Part D, cont'd.



c. Find the weight of each block.

cylinder = _________ pounds

sphere = _________ pounds

cube = _________ pounds

d. List or draw the steps you followed to find the weights of the blocks.

Problem D4.

Start	→	+ 2	→	x 3	→	End

For each Start number, give the End number. For example, when the start number is 5, here's how you get to 21, the End number.

(Start) 5 + 2 = 7; 7 x 3 = 21 (End)

a. Start: 0; End: _________

b. Start: 1; End: _________

c. Start: 3; End: _________

When the End number is 24, the Start number is 6.

For each End number, give the Start number. For example, when the End number is 24, here's how you get to 6, the Start number.

(End) 24 ÷ 3 = 8; 8 - 2 = 6 (Start)

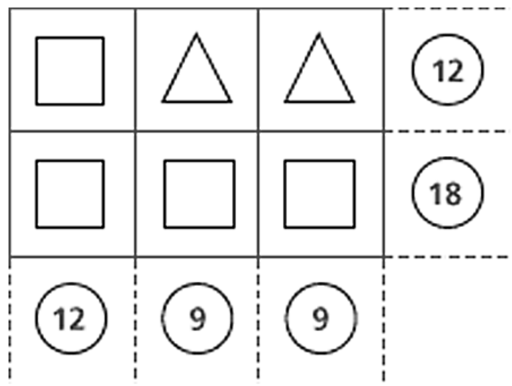
d. End: 12; Start: _________

e. End: 21; Start: _________

f. End: 36; Start: _________

Part D, cont'd.

Problem D5. In the following table, the same shapes are the same numbers. The numbers in the circles are sums, reached by adding across and adding down.



a. What number is the square?

b. What number is the triangle?

c. Explain how you found the numbers.

Problems D1-D5 are taken from Groundworks: Algebraic Thinking, Grade 5, *by Carole Greenes and Carol Findell (New York: Creative Publications, Wright Group/McGraw-Hill, 1998).*

Part E: Critiquing Student Lessons

(20 MINUTES)

In the grades 3-5 curriculum, students are frequently asked to think about patterns, but often their "pattern-sniffing" skills end with simply finding the next object. **[SEE NOTE 8]**

Problem E1.

$\frac{1}{2} \div \frac{1}{8} = 4$	$\frac{1}{2} \times \frac{8}{1} = 4$	$\frac{3}{4} \div \frac{1}{4} = 3$	$\frac{3}{4} \times \frac{4}{1} = 3$
$\frac{5}{6} \div \frac{2}{3} = 1\frac{1}{4}$	$\frac{5}{6} \times \frac{3}{2} = 1\frac{1}{4}$	$\frac{3}{4} \div \frac{3}{8} = 2$	$\frac{3}{4} \times \frac{8}{3} = 2$
$\frac{1}{2} \div \frac{3}{8} = 1\frac{1}{3}$	$\frac{1}{2} \times \boxed{?} = \boxed{?}$	$\frac{1}{6} \div \frac{1}{3} = \frac{1}{2}$	$\frac{1}{6} \times \boxed{?} = \boxed{?}$

a. What algebraic ideas are in this lesson?

b. How are patterns used in this lesson?

c. What mathematics do you think students would learn from this lesson?

d. Are there any misconceptions that students might develop from this lesson?

e. How would you modify the problem, or what additional questions might you ask, to incorporate the framework for analyzing patterns?

NOTE 8. In the lesson on dividing fractions, students might see a pattern of multiplying by the reciprocal. Students should, however, be asked if they understand why this method works. For this particular set of problems, this is a crucial question in their understanding of operations with fractions. Observing the pattern is not enough.

The same questions need to be asked regarding the problem from *Investigations in Number, Data and Space*. One of the key components in analyzing patterns is determining why they continue as they do, even "down the line."

Problem E1 is taken from Addison Wesley Mathematics, Grade 5 *(Menlo Park, Calif.: Scott Foresman-Addison Wesley, 1995).*

Part E, cont'd.

Problem E2.

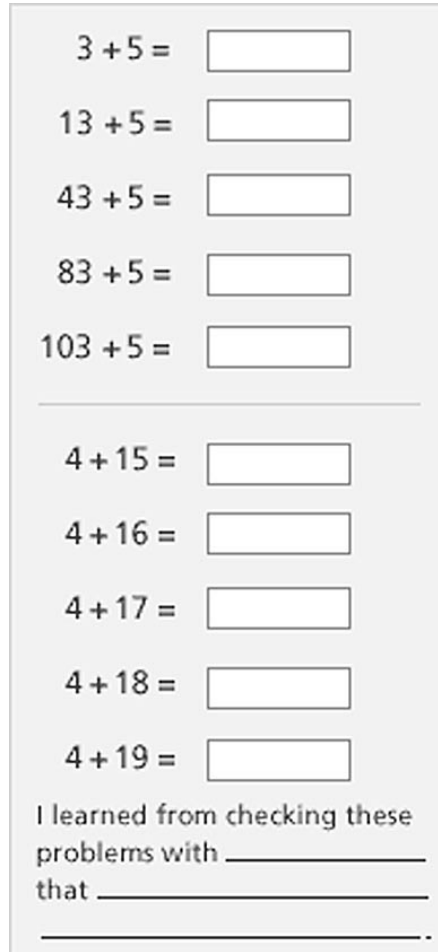
a. What algebraic ideas are in this lesson?

b. How are patterns used in this lesson?

c. What mathematics do you think students would learn from this lesson?

d. Are there any misconceptions that students might develop from this lesson?

e. How would you modify the problem, or what additional questions might you ask, to incorporate the framework for analyzing patterns?



3 + 5 = ☐

13 + 5 = ☐

43 + 5 = ☐

83 + 5 = ☐

103 + 5 = ☐

4 + 15 = ☐

4 + 16 = ☐

4 + 17 = ☐

4 + 18 = ☐

4 + 19 = ☐

I learned from checking these problems with ________ that ________________________.

Problem E2 is taken from Investigations in Number, Data and Space, Grade 4, *by Susan Jo Russell (Menlo Park, Calif.: Scott Foresman-Addison Wesley, 1997), p.14.*

Homework

Problem H1. Interview a teacher in the grade level above you. Pick one of the problems on the previous pages, and ask them the following:

a. How does the content of this problem prepare students for algebraic thinking in their grade?

b. Why do they think this content is important?

c. How could this problem be extended for students in their grade?

Problem H2. Look at a problem in your own mathematics program for your grade level that you think illustrates algebraic thinking. If you were to teach this problem after taking this course, how might you modify or extend it to bring out more of the content of algebraic thinking?

Solutions

Part A: Classroom Video

Problem A1. Reflection on the three questions should include the ideas described below.

a. The fundamental algebraic idea (content) in this video is the notion of function and inverse function as doing and undoing.

b. The teacher expects students to write a function rule to represent a relationship shown in a table of values and then find a rule for the inverse function.

c. Students demonstrate their knowledge of the intended content by writing function rules and by describing inverse functions as reversing a recipe by reversing steps and using "opposite" operations.

Problem A2. Ms. Jones asks students to think of a function as a recipe for "doing," which then leads to inverse functions as "undoing" the same recipe.

Problem A3. The students describe undoing as reversing the steps and using opposite operations.

Problem A4. Ms. Jones could have used recursive thinking with the function machine that doubles and adds one. This function could be described recursively by saying, "The Out number for In = 1 is 3. The Out number for any other In is the previous Out + 2."

Part B: An Example for Developing Algebraic Thinking

Problem B1. Answers will vary. One question might ask how to use the pictures to help find the relationship between the pool number and the numbers of white and blue tiles.

Problem B2. Answers will vary. For the question above, you might discuss building each pool. For example, the blue part of Pool 1 is a 1-by-1 square and takes one blue tile, the blue part of Pool 2 is a 2-by-2 square and takes four blue tiles, etc.

Problem B3. Answers will vary. At this level, many students will see that the blue tiles are always in the shape of a square, and that there are always an even number of white tiles. Some may recognize that the number of white tiles is always a multiple of four.

Problem B4. Answers will vary. At this level, teachers should encourage students to describe the shape of the blue tiles and write a rule to represent the relationship between the pool number and the number of blue tiles. Then ask them to think about putting white tiles around that blue pool. How many white tiles do you need for the top? How many for the bottom? Now think about tiles around the sides. How many white tiles does this require in all?

Problem B5. Answers will vary. At this level, students should actually build the pools with two different color square tiles, then describe what they built. The process of building will help students put the patterns into words and symbols. Many students will use all these skills when solving this problem.

Problem B6. Answers will vary. All of the ideas described in the article's Grades 3-4 section are appropriate for this level student. Many students at this level will be able to answer questions posed in the article's Grades 5-6 section.

Problem B7. Answers will vary. Ask students to look at two consecutive entries and think of ways to get from one to the other. Then have students continue that pattern to see if other entries are correct.

Problem B8. Answers will vary. Often students see a rule to get from one entry to the next. Teachers should help students think of ways to connect each entry to its place in the list.

Solutions, cont'd.

Part C. Patterns That Illustrate Algebraic Thinking

Possible responses for questions (a) through (d), which apply to all the problems in Part C, are as follows:

a. Answers will vary. Most people draw a diagram showing the number of trips to get 1 or 2 adults across the river and then generalize to 8 adults.

b. Answers will vary. The problem requires students to find a way to represent the problem, look for a pattern, and generalize. The problem can be thought of as a recursive pattern.

c. Answers will vary. Most students draw a diagram with arrows showing people crossing the river in each direction. This arrow representation allows students to "see" the number of trips required for each adult.

d. Answers will vary. For example: What sequence of events must happen to get 1 adult across the river? Where are the 2 children at the end of this sequence? How many trips does the sequence require? What must happen so that 1 adult and 2 children are across the river? How does this change if there are 2 adults? Think of "undoing" your sequence to find out the number of adults with 2 children that required 13 trips to cross the river.

Problem C1. The answer is 33 trips: 4 trips to bring over each adult, plus 1 more trip for the 2 children.

Problem C2. The answer is 25 trips: 4 trips to bring over each adult, plus 1 more trip for the 2 children.

Problem C3. The answer is 61 trips: 4 trips to bring over each adult, plus 1 more trip for the 2 children.

Problem C4. The answer is 93 trips: 4 trips to bring over each adult, plus 1 more trip for the 2 children.

Problem C5. The answer is 401 trips: 4 trips to bring over each adult, plus 1 more trip for the 2 children.

Problem C6. The answer is $(4N + 1)$ trips: 4 trips to bring over each adult, plus 1 more trip for the 2 children.

Part D. More Problems That Illustrate Algebraic Thinking

Problem D1. The answers are: You must take out 4 black marbles. The picture should include 6 red and 2 black marbles. The mathematical content is proportional reasoning.

Possible responses to the questions to consider:

a. This prepares students for reasoning with proportions, recognizing equivalent ratios, and using drawings to help solve problems.

b. This content requires the uses of proportional reasoning and all the skills related to patterns: finding, describing, explaining, and using patterns to predict.

c. This problem makes use of proportional reasoning discussed in Session 4 and all the skills related to patterns discussed in Session 2 (finding, describing, explaining, and using patterns to predict).

d. Students approach these problems in several ways:
   - Circle 3 reds and 1 black until there are no more reds, and count the rest
   - Double 3 red and 1 black to get 6 red and 2 black, then subtract
   - Count by threes to get two sets of 3 red, find two sets of 1 black, then subtract
   - Six reds gives $6 \div 3$, or two sets of red; next, find two sets of 1 black, then subtract

Solutions, cont'd.

e. To extend students' understanding, ask questions such as the following:
   - How many red marbles are in the bag? How many black?
   - How many black marbles should be left with 3 red?
   - How could you show that 3 reds match with 1 black?

f. Answers will vary. Very few textbooks contain problems of this type.

Problem D2. For the first set of scales the answer is 1 cube. For the second set of scales the answer is 1 cylinder.

Possible responses to the questions to consider:

a. The mathematical content is balance, the notion that the objects on the lower pan on a pan balance weigh more than the objects on the higher pan, and that one object that balances with two or more objects is the heaviest. This prepares students for understanding equality as balance and the notion that adding or removing the same items from both sides of a balance retains balance.

b. This content introduces equality as balance to set the stage for solving equations by manipulating equations while maintaining balance.

c. This content illustrates methods for solving linear equations.

d. Students approach these problems in several ways. One way is shown for each problem.
   - For the first set of scales: Remove the cube from both sides in Scale B to show that 1 cylinder balances 2 spheres. Then replace each of the cylinders in Scale A with 2 spheres to show that 1 cube balances 6 spheres. There are 6 spheres on Scale C. One cube will balance those 6 spheres.
   - For the second set of scales: Remove 1 cylinder and 1 sphere from both sides in Scale D to show that 1 cube balances 3 spheres. Then replace the 2 cubes in Scale E with 6 spheres to show that 1 cylinder and 1 sphere balance 6 spheres. Remove 1 sphere from each side again to show that 5 spheres balance 1 cylinder. There are 6 spheres on Scale F. One cylinder will balance those 5 spheres.

e. To extend students' understanding, ask them to describe other ways to solve the problems or to make up weights for each block that would preserve the balance, and ask questions such as the following:
   - Are there any blocks that are on both sides of Scale B?
   - What will happen to the balance if you remove 1 cube from each side?
   - Are there any blocks that are on both sides of Scale D?
   - What will happen to the balance if you remove 1 sphere from each side?

f. Answers will vary. Very few textbooks contain problems of this type.

Problem D3. The answer for the first set of scales is: cylinder = 4, sphere = 3, cube = 5. The answer for the second set of scales is: cylinder = 6, sphere = 4, cube = 2

Possible responses to the questions to consider:

a. The mathematical content is variables and solving three equations with three variables.

b. This content prepares students for understanding that they can replace a variable with its value.

c. This content illustrates methods for solving systems of linear equations.

Solutions, cont'd.

d. Students approach these problems in several ways. One way is shown for each problem.

- For the first set of scales: Scale C shows 1 cube and 1 sphere weigh 8 pounds. Replace the cube and sphere on Scale B with 8 pounds. That means the cylinder is 4 pounds (12 – 8). Next look at Scale A. Replace the cylinder with 4 pounds. That means the 2 spheres weigh 6 pounds (10 – 4). Then 1 sphere weighs 3 pounds (6 ÷ 2). From Scale C, if 1 sphere weighs 3 pounds, then the cube weighs 5 pounds (8 – 3).
- For the second set of scales: Scale F shows 1 cylinder and 1 sphere weigh 10 pounds. Replace 1 cylinder and the sphere on Scale D with 10 pounds. That means the remaining cylinder is 6 pounds (16 - 10). Next look at Scale F. Replace the cylinder with 6 pounds. That means the sphere weighs 4 pounds (10 - 6). From Scale E, replace the 2 cylinders with 12 pounds (6 + 6), then the cube weighs 2 pounds (14 – 12).

e. To extend students' understanding, ask questions such as the following:

- Look at all three scales. Do any of the scales contain the same items?
- Which scale would you use first? Why?
- What do Scale C and Scale F tell you?
- How does what Scales C and F show help you with Scales B and E?

f. Answers will vary. Very few textbooks contain problems of this type.

Problem D4. The End numbers for the first set are: 6, 9, 15. The Start numbers for the second set are: 12, 5, 10.

Possible responses to the questions to consider:

a. The mathematical content is function.

b. This content prepares students for understanding function as a sequence of computations.

c. This content illustrates the use of inverse operations.

d. Students approach these problems in several ways. One way is shown here.

- Start with 0: 0 + 2 = 2; 2 x 3 = 6
- Start with 1: 1 + 2 = 3; 3 x 3 = 9
- Start with 3: 3 + 2 = 5; 5 x 3 = 15
- End with 12: 12 ÷ 3 = 4; 4 – 2 = 2
- End with 21: 21 ÷ 3 = 7; 7 – 2 = 5
- End with 36: 36 ÷ 3 = 12; 12 – 2 = 10

e. To extend students' understanding, ask questions such as the following:

- Suppose the start number is 4. Follow the arrows. What do you do first? What is the result? What do you do next? What is the End number?
- Suppose that the End number is 18. How can you find the Start number? Think about going backwards from 18. What number did you multiply by 3 to get 18?

f. Answers will vary. Very few textbooks contain problems of this type.

Solutions, cont'd.

Problem D5. The square is 6 and the triangle is 3.

Possible responses to the questions to consider:

a. The mathematical content is variable.

b. This content prepares students for solving two equations with two variables.

c. This content illustrates the use of division and replacement to solve equations.

d. Students approach these problems in several ways. One way is shown here.
Column 1 shows the sum of two squares is 12. That means that one square is 6 (12 ÷ 2).

   Column 2 shows the sum of one triangle and one square is 9. Replace the square with 6. That means that the triangle is 3 (9 – 6).

e. To extend students understanding, ask questions such as the following:
   - What shapes do you see in the first column?
   - What does the number at the bottom of the first column mean?
   - What number does the square stand for? How do you know?

f. Answers will vary. Very few textbooks contain problems of this type.

Part E: Critiquing Student Lessons

Problem E1.

a. There is no algebra specifically shown in this problem. It does, however, examine some patterns.

b. The patterns used in this problem compare dividing by a fraction and multiplying by its reciprocal.

c. The lesson assumes that students will learn that dividing by a fraction and multiplying by its reciprocal produce the same result.

d. No example shows that dividing by a whole number is the same as multiplying by its reciprocal. This could lead to confusion. There is no attempt to show why the process works.

e. The lesson should include dividing by a whole number, and it should include an explanation of why the process works.

Problem E2.

a. There is no algebra specifically shown in this problem. It does, however, examine some patterns.

b. The patterns used in this problem show two things: a) The units digit of a sum remains the same if the units digits of the addends remain the same; and b) The units digit of a sum increases by one as one addend remains the same and the other addend increases by one.

c. The lesson assumes that students will learn the patterns described above.

d. Students may not notice the patterns, or they may overgeneralize to think that all sets of problems will show similar patterns.

e. The lesson should ask children to generalize and describe the patterns they see.